High-school students' productive struggles during the simplification of trigonometrical expressions and the proving of trigonometrical identities

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DECLARATION

I, Anthony Wilfred Sayster, hereby declare that this dissertation is an original work. It does not contain copied data, tables, graphs, or writings by other authors. Sourced information from other persons has been referenced accordingly.

Signed by candidate 1/26/23

Candidate: Anthony Sayster Date

DEDICATION

I dedicate this thesis to Jesus Christ, who has and will always provide me with the best life.

Proverbs 24
By wisdom a house is built,
and through understanding it is established;
through knowledge its rooms are filled
with rare and beautiful treasures.

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ABSTRACT

This study is an investigation into school students' productive struggles in the *simplification of trigonometric expressions* and *proving of trigonometric identities*. Although studies have been published on the teaching and learning of trigonometrical concepts in schools and teacher education, there is a lack of published research into students' productive struggles in the simplification of trigonometric expressions and proving of trigonometric identities. To fill this gap in the literature, this study used a sample of 16- and 17-year-old students at a rural high school in South Carolina in the United States of America to conduct a study on the use of productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities.

The study used the Anthropological Theory of the Didactic by Chevallard (1992) as the main theoretical framework. However, this main framework was supported by other frameworks. The Anthropological Theory of the Didactic contends that mathematical activities such as simplifying trigonometric expressions and proving trigonometric identities must be interpreted as a human activity rather than seeing these mathematical activities as a language, the creation of concepts (for example, "simplification" or "proof") or a cognitive process. A praxeology consists of two parts, namely (in Greek) the *praxis,* or "know how", and the *logos,* or "know why". The praxis is commonly known as the *practical block,* and the logos the *theoretical block.* This means that the Anthropological Theory of the Didactic can be used to describe *how* certain actions regarding the simplification of trigonometric expressions and proving of trigonometric identities take place, and *why* these actions take place.

The exercises in the activities were obtained and adapted from the students' prescribed textbook. These activity questions were sequenced using the Development Cognitive Abilities Test (DCAT). The DCAT reflects Bloom's (1956) hierarchy of cognitive abilities.
This means that the exercises were organised in three groups of increasing complexity, i.e., easy, medium, and difficult. The easy exercises related to the DCAT's *Basic Cognitive Abilities,*

referred to as DCAT 1; the medium exercises related to *Application Abilities*, referred to as DCAT 2; and the difficult exercises related to *Critical Thinking Abilities*, referred to as DCAT 3. The data in this study consists of video recordings from classroom observations in real time transcribed verbatim, documentary analysis of students' assessments, and audio-recorded focus group interviews. The focus group interviews were also transcribed verbatim. Each transcription focused on a different aspect of the students' productive struggles in the simplification of trigonometric expression and the proving of trigonometric identities. Errors made by the students in written assessments were analysed using the Newman Error Analysis framework. By using Newman Error Analysis, this study could investigate and compare how the errors on the assessments were related to the students' struggles as observed during the teaching and learning of the activity questions. Due to Co-Vid 19 restrictions that resulted in logistical difficulties, only one class of 15 students participated in this study.

After listening to the focus group recordings numerous times and reading the transcripts, common patterns were noted that had emerged, either from paraphrasing or from direct quotes.

The primary research question is:
What is the nature of the productive struggles experienced by high-school students during the *simplification of trigonometric expressions* and *proving of trigonometric identities* and how do these productive struggles influence the learning and teaching of trigonometry?

The study findings were that the students struggled with "carrying out known mathematical processes" such as manipulating equations, knowing under what conditions cancellation of terms can be applied, adding and subtracting algebraic fractions involving trigonometric expressions, and factorisation of trigonometric expressions. In addition, there were misconceptions about the concept of "simplification". Delayed impasse struggles occurred; this is when a student does not initially struggle to get started with a question, but the struggling becomes apparent as the student progresses with the question.

The students committed fewer Newman errors in proving trigonometric identities than in the simplification of trigonometric expressions. Subsequently, students performed better at proving trigonometric identities than at simplifying trigonometric expressions. It could well be that through productive struggles, the students developed some of their own strategies from the simplification of trigonometric expressions. Alternatively, proving identities could be seen as "easier", since the students already know what the "answer" should be. Nonetheless, students still struggled to carry out common mathematical processes such as factorisation and the manipulation of algebraic fractions. Regarding factorisation and manipulation of algebraic fractions, students compartmentalised knowledge. For example, most students knew how to factorise algebraic expressions, but failed to see the resemblance between algebraic expressions and trigonometric expressions (and consequently, how to factorise trigonometric expressions).

Although there was a decrease in the number of Newman errors from the simplification of trigonometric expressions to the proving of trigonometric identities, there was an increase at the comprehension hierarchy, which may be attributed to the fact that the students might have struggled with the concept of "proof". Additionally, students in this study struggled with the concept of "simpler". Some students thought that the solution to a simplification question should be more complex than the original question. Nonetheless, with both the simplification of trigonometric expressions and the proving of trigonometric identities it remained a challenge for the students to apply prior knowledge in a new mathematical context such as trigonometry.

The significance of the study's findings is that they suggest that teachers re-evaluate how to instruct known mathematical processes and procedures, so as not to compartmentalise mathematical knowledge.

Productive struggles may not always produce correct answers; but given sufficient time and appropriate intervention by their teacher, students can build their own knowledge and become independent thinkers who can apply prior knowledge in new contexts such as the simplification of trigonometric expressions and the proving of trigonometric identities.

In future research, productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities should be explored with a bigger, more diverse group of students, taught by more than one teacher at more than one school. In addition, to investigate the long-term effects of productive struggles a study lasting more than six months could be carried out.

DEFINITION OF TERMS

Anthropological Theory of the Didactic: theoretical framework designed by Chevallard (1992) to describe any human activity, for example the simplification of trigonometric expressions and proving of trigonometric identities.

Newman Error Analysis: the framework used to analyse errors made by students on their assessments.

Praxeologies: used by the Anthropological Theory of the Didactic to describe human activity.

Prior knowledge: the knowledge that students had gained up until the beginning of this study.

Productive struggles: an "effort to make sense of mathematics, to figure something out that is not immediately apparent" – Hiebert and Grouws (2007), p.287.

Proving of trigonometric identities: showing that two trigonometric identities are equivalent.

Simplification of trigonometric expressions: the process of making any trigonometric expression "easier".

Trigonometry: the branch of mathematics that deals with the relationship between the sides and angles of a triangle.

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CHAPTER ONE: INTRODUCTION

1.1 Introduction

This study investigates high-school students' productive struggles during the simplification of trigonometric expressions and proving of trigonometric identities. The participating students were a group of 16- and 17-year-old high-school students. The study took place at a small rural school in South Carolina, in the south-east of the United States. This group of students represents the top echelon of the mathematics student body at the school.

Chapter One gives an overview of the initiation of this study. First there is a summary of the background to this study, the students' productive (and unproductive) struggles, the influence of the teacher on students' struggles, and why trigonometry was chosen. Next, active and passive learning are juxtaposed, and a justification is given for the use of active learning in this study. Following this, the statement of the problem is presented, and a rationale for this study is given. The essence of this study (that is, the research questions) is stated, followed by how this study will contribute to the body of knowledge. Lastly, because of extraordinary circumstances due to the COVID-19 pandemic that took place during this study, the reader is made aware of the limitations of this study and how the pilot study guided it.

1.2 Background to the study

This section outlines the background to the students' productive struggles in mathematics, the influence of the teachers on the productive struggles of students, why trigonometry is an important branch of mathematics, and the advantages of active versus passive learning.

1.2.1 Students' productive struggles in mathematics

The National Council of Teachers of Mathematics (NCTM) (2014) defined productive struggles as "struggling at times with mathematics tasks but knowing that breakthroughs often emerge from confusion and struggle" (p.52). In the context of this study, the 'struggling' occurred when students were presented with activities requiring them to simplify trigonometric expressions and prove trigonometric identities. In their investigations into what quality education means and under what conditions quality education is possible for all learners, Murdoch et al. (2020) described productive struggles as an indicator of quality education. By contrast, Warshauer (2015) called a struggle by students 'unproductive' when they show no signs of accomplishing a mathematical task.

However, the extent to which students struggle to solve trigonometric expressions and prove trigonometric identities can also be attributed to their prior knowledge or lack of it (Maglipong et al., 2015). For instance, how well do the students understand fractions? Are the students fluent in all types of quadratic factorisation methods? Are they used to struggling, or were they taught in a very procedural fashion? This prior knowledge also contributes to students' belief regarding how mathematics is learned. For example, some students believe that prior to attempting a mathematical exercise, the teacher should explain similar examples. Similarly, the belief of teachers about how mathematics should be learned influences how they teach mathematics. For example, some teachers believe the instructor should be the focal point of new knowledge and innovative strategies to solve exercises in the classroom. Also, some teachers see struggling in the classroom as a reflection on their instruction. That is, they would rather let the students do exercises with predictable answers than give them challenging exercises to struggle with. In the NCTM publication *Principles to Action*, these beliefs are categorised as being both productive and unproductive (NCTM, 2014, p.11); they are listed in Table 1.1 below.

Table 1.1: The NCTM's list of teacher's belief about teaching and learning (NCTM, 2014, p.11)

Unproductive beliefs	Productive beliefs
Mathematics learning should focus on practising procedures and memorising basic number combinations.	Mathematics learning should focus on developing an *understanding* of procedures and concepts through problem solving.

Students should only learn and use *the same* standard computational algorithms and the same procedures to solve questions.	All students should have a *range of* problem-solving strategies including but not limited to computational algorithms and standard procedures.
Students can only learn to *apply* mathematics after they have *grasped basic skills.*	Students can learn mathematics through *exploring* and solving *contextual mathematical problems.*
The teacher is the *centre of knowledge* in the classroom.	The role of the teacher is to *engage students* in meaningful mathematics that promotes reasoning and problem-solving skills.
The role of the student is to *memorise* procedures to solve questions and use them to solve similar questions in their homework and assessments.	The role of the student is to be actively involved in sense-making of mathematics by using *varied strategies* to solve challenging questions and making connections with prior knowledge.
An effective teacher *guides* students though *step-by-step questions* to avoid confusion.	An effective teacher provides students with *appropriate challenging* questions, and encourages perseverance in their struggles.

In addition, the NCTM's principles necessitate an ideological shift when it comes to the construct of 'productive struggle'. To struggle is not easy; however, solving predictable questions is not conducive to effective learning. Thus, both teachers and students should reconsider what it is to be effective teachers and learners. One outcome of productive struggle is creativity, which Roble (2017) agrees comes about through experimentation. For example, by experimenting, a student might discover that a trigonometric identity can be proved by starting on either side of an equation.

Although it seems counterintuitive to introduce *productive failure* as a blueprint for *productive struggle*, Kapur (2010) did precisely this when he juxtaposed *direct instruction* and *productive failure.* Direct instruction is the process of teaching concepts and procedures for solving specific types of problems, and then assessing students on how well they employ those procedures and concepts (Luke, 2014b). Kilpatrick et al. (2001) refers to this seamless execution of procedures to solve mathematical problems successfully as *procedural fluency.* For instance, a teacher might explain the procedure for solving a quadratic equation using the quadratic formula, and then assess the students on their ability to solve similar questions applying the quadratic formula.

The converse of this process – that is, when students first attempt to problem-solve but reach an impasse, and then are taught the concepts afterwards – was called *productive failure* by Kapur (2010) and Kapur and Rummel (2012). Kapur and Bielaczyc (2012) argued that failure to solve a problem successfully using prior knowledge may be productive, in the sense that students may be better prepared to learn from direct instruction afterwards. The authors thus promoted the notion of students "developing deeper understandings" (Kapur & Bielaczyc, 2012, p.46) when they engage in problem-solving through struggle, even when that struggle may be unsuccessful. Additionally, Granberg and Olsson (2015) suggested that students who fail to solve a question on their own are more likely to examine their methodology, compared to students who attempt to solve their questions procedurally.

During the process of struggling to figure out a solution, students may also inadvertently create their own, more efficient methods (Granberg, 2016). Moreover, students who generate their own solution to a question – even if that solution is strewn with errors – outperform procedurally instructed students on tests, as reported by Granberg (2016). Similarly, Jonsson et al. (2014) reported that when comparing students who created their own knowledge with students that were taught in a traditional way, the students who created their own knowledge performed significantly better on post-tests. In addition, Kapur (2011; 2014) reaffirmed the importance of struggling by reporting that students who create their own solutions through struggling outperform procedurally instructed students on post-tests, even if they do not solve the question. Furthermore, young children who experienced productive struggles when solving questions prior to instruction benefited from increased conceptual understanding (DeCaro & Rittle-Johnson, 2012).

The construct of *productive struggle* proposed by Hiebert and Grouws (2007) as an "effort to make sense of mathematics, to figure something out that is not immediately apparent" (p.287) was used in this study to describe the actions of the students who were tasked with simplifying trigonometric expressions and proving trigonometric identities.

DeJarnette (2014) extended this *productive struggle* construct when she analysed how students learn when they interact with peers in a computer programming environment. She used an environment called 'Etoys' to examine the students' interaction with one another when dealing with trigonometric problems involving sine and cosine. She coined the term *collaborative struggle* for the struggle that the students experienced when dealing with Etoys. This is defined as a struggle that occurs between students when they collaborate (in pairs) to make sense of mathematics. DeJarnette (2014) showed how students developed an understanding of the sine and cosine function through collaborative struggle.

Productive struggle requires perseverance; and "perseverance is only useful when students have sufficient prior knowledge and metacognitive abilities to make their struggles potentially productive" (Star, 2015, p.9). A lack of prior knowledge might lead to what is described in Chapter Three as occupying the 'working memory' with disorder. This disorder might entail looking for solutions in ways such as trial-and-error methods, thus making the working memory (which is necessary for learning) less prone to learning new methodologies, as proposed by Kirschner et al. (2006). For students to experience productive struggle when simplifying trigonometric expressions and proving trigonometric identities, the exercises they perform must be well designed, so as to be within reach of their cognitive abilities. That is, Granberg (2016) argues, students should have the innate ability to realise that a question is solvable – even if their prior knowledge may need to be re-examined if it is flawed. Moreover, researchers have suggested that in fact, students become aware of these deficiencies in their prior knowledge and are likely to be focused on these shortcomings during instruction (Glogger et al., 2013). For example, during the simplification of trigonometric expressions, a student might realise that their understanding of adding fractions is not up to the task, and would then solicit help from the teacher to re-explain this concept. Such assistance from the teacher, which does not always eliminate the productive struggle, is a crucial element of the whole struggle process (Kapur, 2015). However, struggling as a way to deepen understanding of concepts is not yet accepted by the mathematics education community at large. In fact, Livy et al. (2018) contended that teachers are used to progress that is effortless. However, Daily (2021) argued that constantly assisting struggling students can be detrimental to their struggle. Productive struggle requires time and effort. Thus, interfering in this process can diminish the student's ability to build critical thinking skills (Daily, 2021).

The NCTM (2014) warned that the notion of struggle does not fit into the model of what an 'ideal' classroom should look like – that is, students quietly working on procedural exercises with predictable answers. In this scenario, the teacher is the source of knowledge, and emphasis is placed on algorithms and procedures for solving questions. For example, when discussing factorisation methods with the students, the teacher might explain the process of finding the greatest common factor. For many teachers, the collaboration and discussion between students and teachers that often goes hand in hand with struggling in the classroom is perceived as creating disciplinary problems. Teachers try to eradicate 'disruption' in the classroom by giving the students step-by-step procedures for solving questions. However, the NCTM (2014) classifies this way of teaching and learning as "unproductive" (p.15). In her investigation into how productive struggles can be an effective strategy in elementary school mathematics classrooms, Daily (2021) appropriately posed the question: "What is to be gained from this uncomfortable and time-consuming method?" (p.85).

The process of struggling in mathematics may have long-term advantages for students, as reported by Kapur (2011) and DeCaro and Rittle-Johnson (2012). Struggling in mathematics has the further benefit of long-term retention, according to Kapur (2010). Moreover, O'Dell (2018) contended that students feel proud when they are successful in their struggles. Livy et al. (2018) stated that if students fail, yet still want to continue with difficult mathematical questions, then their productive struggles have been successful.

Struggling with mathematics in the classroom can have other advantages for students; for example, it may mean that no homework is issued. Bempechat et al. (2011) contended that students favour difficult classes in which 'homework' is done in class. Also, Ambrose et al. (2010) reported that students show a liking for classes that dispel any fear of giving 'wrong' answers. Subsequently, Bempechat et al. (2011) showed that there is an association between high achievement and the enjoyment of a subject. Building on this positive outcome of subject enjoyment, Else-Quest et al. (2010) showed that students with greater confidence in their mathematical ability are more likely to take more mathematics courses. In addition, Ahmed et al. (2013) found a link between loss of confidence and performing below one's potential. However, to date there has been no significant research on student-centred learning through exploring students' productive struggles in trigonometry – in terms of both simplification of trigonometric expressions and proving trigonometric identities. Therefore, this study is attempting to fill that void by exploring students' productive struggles to simplify trigonometric expressions and to prove trigonometric identities.

1.2.2 The influence of teachers on the struggles of students

The NCTM (2014) suggested a way in which teachers can adapt their teaching approach to help solve this perceived 'problem' of struggle in the mathematics classroom. According to the NCTM (2014), "Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and support to engage in productive struggle as they grapple with mathematical ideas and relationships" (p.48). Moreover, the NCTM (2014) advises teachers to issue mathematical questions to solve that are within the reach of students, in order to invoke productive beliefs about learning.

Not all struggles in mathematics are productive. Allowing 12-year-old students to struggle with an advanced calculus question would most probably be fruitless. Although the students would indeed struggle, their chances of solving such a problem would be low and would just lead to what Beilock et al. (2010) reported as frustration and self-doubt. These researchers concluded that such frustration and self-doubt often affect students' reasoning ability. However, Sun and Van Es (2015) stated that if teachers adapt their teaching through noticing (in the case of this study) productive struggles, the mathematical abilities of their students may improve. Furthermore, Star et al. (2011) contended that teachers who use noticing for improving their teaching are able to make classrooms more conducive to learning.

Vygotsky (1978) asserted that learning takes place in the *zone of proximal development*, where students can get help from a peer or the teacher, but are allowed to solve a question by themselves. Thus, teacher noticing plays an important role in dealing with productive struggles; through noticing, the teacher must identify the zone of proximal development so that he or she does not give the student too much or too little assistance. This study uses Sherin et al. (2011) definition of teacher noticing in mathematics as managing the classroom information during instruction.
The zone of proximal development is an ideal space for the teacher to experiment with what the NCTM (2014) referred to as 'productive beliefs', as discussed in Table 1.1.

The constructs of teacher questioning and noticing will be further elaborated on in Chapter Two.

1.2.3 Why trigonometry?

Trigonometry is a branch of mathematics that links algebra, geometry and graphical reasoning, according to the National Governors Association Centre for Best Practices (NGAC, 2010). In line with Akkoc's (2008) contention that trigonometry is an area of mathematics education that does not receive enough research, this study expanded on this idea by exploring the productive struggles that students experience when simplifying trigonometric expressions and proving trigonometric identities. Indeed, trigonometry is not the best-loved mathematics topic of the majority of high-school students; most find it abstract and very hard to understand (Mensah, 2017). This is also the case at the school that participated in this study. In order to understand why trigonometry matters, students must understand trigonometry's context in mathematics (Gür, 2009). Recent research on the subject has focused on the learning and teaching of trigonometry and trigonometric functions using (as an example) computers (Curri, 2012; Hertel & Cullen, 2011; Lotfi & Mafi, 2012). Also, motivated by most students' dislike of trigonometry, Gerhana, Mardiyana and Pramudya (2017) proposed the project-based learning model for teaching trigonometry. Like this study, project-based learning offers students the opportunity to collaborate in groups to solve a mathematics question. Bressoud (2010) advocated that trigonometry should be taught in a way that promotes deeper understanding.

The American College Testing (ACT) programme's 2016 report *The Condition of College and Career Readiness* shows that only 25% of South Carolina (which is where this study was conducted) students who sat for the college-entrance examination tested as career ready, compared to a US national average of 41%. Questions on trigonometry make up 7% of the 60 multiple-choice questions in the mathematics portion of the ACT test; although trigonometry forms the smallest portion of the test, it is nonetheless an integral part of it. A worrisome 2020 report by the South Carolina Education Oversight Committee states that South Carolina students' ACT results (including mathematics and thus trigonometry) rank near the bottom among the 50 states and the District of Columbia. The director of the South Carolina Educational Oversight committee stated that: "These ACT scores emphasise our state has to improve the teaching and learning of reading and mathematics" (paragraph 4). Furthermore, the difficult mathematics questions on the ACT test are not related to advanced calculus, but are insightful questioning of the students' prior knowledge of basic mathematics. This would suggest that mathematics should be taught in such a way that students develop a deep understanding of the questions presented to them.

For the novice or inexperienced teacher, trigonometry can be very challenging. Dündar and Yaman (2015) found that preservice mathematics teachers' conceptual knowledge was lacking, despite their having high procedural trigonometry knowledge. The textbook by Demana et al. (2015) adopted by the participating school introduces trigonometry to students via the 'problem of angular measure', by discussing why radians rather than degrees are best suited to trigonometry. This discussion is followed by some real-life examples of angular and linear motion which may be difficult to understand if one is not practically inclined. The exercises can be off-putting, especially when dealing with (for example) the angular speed of ship propellers or car wheels, or the influence of car wheels on vehicle speed. It is easy to understand why the participating teacher would rather introduce trigonometry by starting with the six trigonometric ratios defined in a right triangle.

At the end of the trigonometry chapter, Demana et al. (2015) conclude with the simplification of trigonometric expressions and the proving of trigonometric identities. But while trigonometry is an important part of the high-school mathematics curriculum, little research has been done on productive struggles relating to these tasks; ‘simplification’, as a concept, is not straightforward. The Cambridge English dictionary defines the word ‘simplification’ as ‘making something easier or less complicated to understand’ but Delice (2002) contended that “simplification is an odd term to use with trigonometric expression” (p.19). Delice (2002) was referring to the fact that it is sometimes hard to distinguish what is ‘simpler’, for example if one is given the choice between $\sin(-x)$ and $-sinx$. Consequently, this study assumes ‘simplification’ to have the lexical meaning of ‘the process of making something easier or less complicated to understand’.

It is hoped that this study will help teachers create new ideas for teaching trigonometry. For example, Irawan et al. (2019) developed trigonometry visualisation concepts to help vocational high school students with their understanding of trigonometry. Putri et al. (2020) used a learning model to better prepare students in using trigonometry for 21st-century skills.

1.2.4 Active learning versus passive learning

Active learning incorporates activities and (as this study does) requires students to think critically when solving non-standard questions, thus enhancing student learning through more content knowledge (Kilgo et al., 2015; Kitchens et al. 2018). In contrast, passive learning promotes the teacher as the single source of knowledge in the classroom (Huggins & Stamatel, 2015; Topcu & Abrahams, 2018). This type of learning where the teacher is the centre of knowledge, teaching algorithms and procedures to solve questions, still seems to be the norm in American high schools, even in what Schoenfeld (1992) calls “good” classrooms. To make matters worse, Porter et al. (1988) found that most mathematics teachers focus on developing procedural and computational skills. This procedural teaching was also evident at the school participating in this study, revealed in interviews with some of the mathematics teachers. Boaler (1998) found that while students can solve procedural mathematics questions, this does not guarantee that they can do mathematics. In this regard, Molina and Sales (2008) recognised the usefulness of employing acquired information in new applications and the importance of instilling these skills in all school disciplines, not only mathematics.

Thus, to accommodate the malleable application of mathematics, students should be taught in a way that makes this transfer of mathematics to new environments come naturally. Boston and Wilhelm (2015) and Stigler and Hiebert (1999) acknowledge that conventional teaching methods (passive learning) do not give students enough practice in mathematical reasoning; rather, they promote procedural learning, followed by exercises that reinforce the procedures. But struggle is essential for intellectual growth, and doubt and confusion create opportunities for deep understanding of mathematical concepts, such as the simplification of trigonometric expressions and the proving of trigonometric identities (Kapur, 2010). Moreover, Herreid (2013) argues that students gain knowledge best when they are active learners; for example, going through productive struggles in trigonometry. In fact, educational researchers from the 1990s to the present have advocated active learning rather than the traditional 'teach in the classroom and practise at home' model (Bergmann & Sams, 2012; Berrett, 2012; Fulton, 2012).

1.3 Statement of the problem

Mathematics – specifically, trigonometry – is taught in a very procedural way at most schools; and most teachers reject the notion of struggling in the classroom (NCTM, 2014). Gür (2009) also contended that trigonometry is a branch of mathematics that students struggle with more than any other branch of mathematics. However, Boston and Wilhelm (2015) and Stigler and Hiebert (1999) contended that this procedural methodology of teaching does not promote insightful learning.

The ACT results for South Carolina students were mirrored at the participating school. This means that approximately 25% of the participating school's student body who sat for the ACT test tested as career-ready. Nonetheless, of interest is the big discrepancy between the students' average high-school mathematics scores and their corresponding ACT test results. This possibly suggests that the teaching methodology at the high school favours procedural learning rather than learning for long-term retention.

However, Roble (2017) contended that productive struggles generate originality. And importantly, in a society that places great value on standardised tests, Kapur (2011; 2014) reported that students who adopt the notion of struggling, although not necessarily productive struggling, outperform procedurally instructed students on post-tests. However, for a student to adopt a struggling mindset might not be easy since struggling might have a negative connotation.

Furthermore, Granberg (2016) reaffirmed the benefits of struggle by contending that students who struggle outperform procedurally instructed students on tests. Consequently, to promote insightful teaching and learning of trigonometry as a topic, this study explores how students' productive struggles can be supported during the learning activities for simplifying trigonometric expressions and proving trigonometric identities.

1.4 Rationale of the study

Previous studies on students' productive struggles have focused primarily on (and have been limited to) examining whether such struggles have occurred, rather than investigating the nature of student struggles in detail (Inagaki et al., 1998; Santagata, 2005). To make up for this shortcoming, Warshauer (2011; 2014a) investigated the nature of productive struggle in middle-school classrooms in detail. Warshauer (2014a) stated that future studies need to build on the descriptions of struggle in different classroom settings, and to ask to what extent task selection contributes to or hinders students' productive struggles. This study wishes to extend Warshauer's (2014a) investigation on productive struggle in a middle-school setting by examining how it may help students in simplifying trigonometric expressions and proving trigonometric identities. Moreover, it challenges the notion that struggling in mathematics is negative and a weakness, as contended by Pasquale (2016). Through skilful teacher questioning and noticing, this study will also shed some light on the level of content knowledge a teacher must possess to promote the learning of trigonometry, thus contributing to Walsh, Fitzmaurice and Donoghue's (2017) investigation of pre-service teachers' trigonometric knowledge. Furthermore, the South Carolina College- and Career-Ready Standards for Mathematics (SCCCR) state that a mathematically literate student should "use critical thinking skills to justify mathematical reasoning and critique the reasoning of others", and "use a variety of mathematical tools effectively and strategically" (Scandrol, 2017, p.138). It is precisely this dilemma that this research wishes to illuminate and begin to resolve, through the exploration of students' productive struggles and the introduction of some innovative learning strategies. What is intriguing about this study is that there was a huge discrepancy between the participating school's advanced mathematics class-average grade, and their score on the mathematics portion of the ACT. Eleven participating students sat for the ACT exam during February 2019. Their average score on the 'Preparing for Higher Mathematics' portion of the ACT exam, which included trigonometry questions, was 49%. The same students' high-school average for mathematics was 96%.

1.5 Aims of the study

As a prelude to the research questions, the reader is reminded that the aim of this study was to investigate high-school students' productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities, and how these struggles can be used to improve the learning and teaching of trigonometry at high-school level.

1.6 Research Questions

Primary research question: What is the nature of the productive struggles experienced by high-school students during the *simplification of trigonometric expressions* and *proving of trigonometric identities* and how do these productive struggles influence the learning and teaching of trigonometry?

The primary research question is broken down into secondary research questions. These are:

(a) What types of productive struggles are observed during the *simplification of trigonometric expressions* and the *proving of trigonometric identities*?

(b) What types of productive struggles from the learning activities remained unresolved during the assessments of simplifying trigonometric expressions and proving trigonometric identities?

(c) What are the qualitative differences in student performance between *simplification of trigonometric expressions* and *proving of trigonometric identities*?

In essence, this study sought to identify the type of productive struggles that remained throughout the investigation as well as to explore whether students performed better when they simplified trigonometric expressions or when they proved trigonometric identities.

1.7 Layout of dissertation

This section presents an overview of the layout of this dissertation. It starts in Chapter One by explaining to the reader what the study is about. The study investigates the simplification of trigonometric expressions and the proving of trigonometric identities. To this end, this section sketches the background to and the motivation for this study. Additionally, this section presents a case for why productive struggle, as a learning method, was chosen, and also presents the research questions on which this study was based.

In Chapter Two this study presents and discusses the theoretical and conceptual resources framing this thesis and reviews the existing literature, including previous empirical research that is relevant to this study. Chapter Two also presents the different frameworks that are used for data analysis in Chapters Four and Five, as well as how productive struggle is framed regarding the main conceptual framework.

Chapter Three outlines the research design for this study, including the methodological choices made, the data collection tools and the data analysis processes employed. The research instruments that formed the backbone of this study, as well as the research procedures, are laid out here.

Chapter Four presents the results and accompanying discussions of the activities involved in the *simplification of trigonometric expressions*.

Chapter Five follows the same pattern as Chapter Four. However, in this chapter the results from the activities involved in *proving trigonometric identities* were presented.

This study concludes with Chapter Six. In this chapter the research questions are addressed, based on the data analysis performed in Chapters Four and Five. Moreover, the implications of the results obtained and recommendations for further study are also discussed.

1.8 Summary

Chapter One explained the premise of the study, an investigation of productive struggle in the simplification of trigonometric expressions and the proving of trigonometric identities.
The reason for using the simplification of trigonometric expressions and the proving of trigonometric identities was explained by highlighting the difficulties that students in high school experience with trigonometry. The research questions were presented here, as well as how this study could contribute to the learning and teaching of mathematics locally and internationally. In Chapter One, this study briefly explained how data were obtained and analysed regarding the simplification of trigonometric expressions and the proving of trigonometric identities.

Chapter Two lays the foundation for this study. This chapter presents the theoretical framework that informed this study, as well as the literature and theory that necessitated this framework. Additionally, in Chapter Two this study introduces a test that is used to inform the activity questions that were used in the simplification of trigonometric expressions and the proving of trigonometric identities.

CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

This chapter presents and discusses the theoretical and conceptual resources framing the thesis, and reviews existing literature that is relevant to the study, including previous empirical research. This study is foregrounded in the Anthropological Theory of the Didactic (ATD). Consequently, this chapter begins with a general view of ATD, then discusses the role of ATD in mathematics education and gives an illustration of how ATD is relevant to this research. Then the chapter introduces the Developing Cognitive Abilities Test (DCAT) and how it was used to inform the cognitive levels of the activity questions used in the simplification of trigonometric expressions and the proving of trigonometric identities.

Next, the chapter explores how students learn, and explains why this study chose productive struggles as the best way to do so. It then presents the type of productive struggles that students may experience, and how a teacher may respond to a productive or unproductive struggle. The chapter then presents how productive struggle positions itself in the ATD. The importance of Mathematical Proficiency is presented, and how success in productive struggles depends on it. An illustration of how mathematical proficiency is situated in the ATD follows.

In conjunction with how students learn, this chapter next introduces the framework for how to Teach for Robust Understanding (TRU), and how TRU positions itself in the ATD. The didactical focus then shifts to Teacher Noticing, the interaction that occurs between teacher and student during teacher noticing, and the position of teacher noticing within the ATD. Teacher Questioning is introduced next, and this chapter describes how this study used Boaler and Brodie's (2004) nine types of questioning for analysing how a teacher might question for deeper understanding. As with the other constructs introduced in Chapter Two, teacher questioning is positioned in the ATD.
In the subsequent sections of this chapter this study shows how these frameworks – that is, the frameworks for TRU, Teacher Noticing and Teacher Questioning – relate to students' productive struggles.

Lastly, Newman Error Analysis (NEA) is introduced, with an explanation of its appropriateness for use in this study. By using NEA on the assessments which took place after the activities, this study could ascertain which productive struggles remained unresolved during the assessments. Chapter Two concludes with how NEA and all the other constructs introduced in this chapter are positioned in the ATD.

2.2 Theoretical frameworks

2.2.1 General view of the Anthropological Theory of the Didactic

In this section, this study discusses the general view of the ATD. In 1991, the French mathematician Chevallard introduced the *Théorie Anthropologique du Didactique,* known in English as the Anthropological Theory of the Didactic or ATD. ATD is a theory based on epistemology (how humans acquire knowledge) that can be used to evaluate mathematics as a subject; that is, how the subject is taught and learned, from the perspective of both the teacher and the student (Chevallard, 1992).

However, ATD can be applied to any variation of human action, not only mathematics. It is chiefly utilised by means of *praxeologies,* or organisations that describe the practices of teacher and student (Barbé et al., 2005). A praxeology consists of two parts, namely (in Greek) the *praxis,* or 'know how' and the *logos*, or 'know why'. The praxis is commonly known as the *practical block*, and the logos the *theoretical block.* The praxis component of a praxeology consists of the *type of task* and the *technique* used to solve that type of task. The ATD's task type in this study refers to the activity questions that evaluate students' resolve to simplify trigonometric expressions and prove trigonometric identities. The participating students needed one or more techniques to solve these problems, such as factorisation, adding trigonometric fractions, cancellation, simplification of trigonometric expressions, and trigonometric identities.

The logos component of a praxeology consists of *technology*, which justifies the *technique* used; and theory, which in turn justifies the technology. The technology for using the techniques described in the previous paragraph is (in the case of simplification of trigonometric expressions) to make the questions 'simpler' or 'easier', and (in the case of proving trigonometric identities) to prove that two identities are equal. The theory used is the application of the trigonometric identities ($sin^2x + cos^2x=1$ and $tanx = \frac{sinx}{cosx}$) in conjunction with basic algebraic processes. The type of task, technique, technology and theory are interdependent.

Chevallard (2015) contended that ATD defines 'didactics' as the scientific approach to conditions of distribution of knowledge to persons and within institutions. In the context of this study, the word 'didactic' refers to the teaching of trigonometric concepts to students at the school where the study took place. More generally, ATD perceives any science – including mathematics – as investigating conditions that have a bearing on human life and its environments (Chevallard et al., 2015).

Although ATD has many applications in mathematics education, as a construct it can be applied in multiple other disciplines. To make science or disciplinary knowledge more teachable, the term *didactic transposition* has been introduced to adapt knowledge to make the information easily 'processable' for students (Achiam, 2014).

The theory of didactic transposition is based on the contention that with a few exceptions, entities of knowledge are designed to be used rather than taught (Kang & Kilpatrick, 1992). Chevallard (1988) described didactic transposition of knowledge as the transfer of knowledge from an entity that can be used as a tool (for example, mathematics as a research tool) to an entity that should be taught and learned (for example, high-school mathematics). In the context of this study, although trigonometry can be applied as a research tool, for example in *Lorentzian geometry,* this type of application of trigonometry is beyond the scope of the average high-school student. Thus, research on trigonometry must be transposed to a construct more understandable to high-school students. And in the case of this study, the participating students were introduced to only two trigonometric identities.

Any adjustment of knowledge for instructional purposes can be labelled a didactic transposition; for example, a teacher might explain the simplification of algebraic expressions in order to better explain the simplification of trigonometric expressions. Chevallard (1988) contended that the difference between teachable knowledge and usable knowledge is that usable knowledge needs no justification for any social purpose – that is, if knowledge is uniquely used in an activity, there is no need to give reasons for using that knowledge to make sense of your activity.

However, Chevallard (1988) also posited that teachable knowledge needs social acknowledgement and legitimisation. This socially acknowledged and legitimate knowledge is derived from scholarly knowledge. Chevallard contended that scholarly knowledge has a binary function; it is knowledge used to produce new knowledge, but also to organise the newly produced knowledge into a logical and consistent theoretical collection. Furthermore, the mechanism of knowledge production and assembly is invisible when teaching. Didactic transposition theory supports the existence of a mechanism that moves from scholarly knowledge to knowledge taught at a high school (Chevallard, 1988).

Achiam (2014) used the construct of didactic transposition to describe different stages of science education. For example, she explained how a research palaeontologist reconstructed a fossil for accurate classification. The palaeontologist transposed knowledge of a two-dimensional fossil structure to describe a three-dimensional image. In the context of this study, He (2019) investigated trigonometric identities at a research level. This type of research-level knowledge is what Chevallard (1988) refers to as 'useable' knowledge. However, the trigonometric identities used in this study have been transposed to be more applicable at high-school level.

Brousseau (2006) contended that "didactic transposition [...] is at the same time inevitable, necessary and, in a sense, regrettable. It must be kept under surveillance" (p.21). This means for example that knowledge will *inevitably* undergo changes from scholarly research knowledge to high-school knowledge. This transformation of knowledge is *necessary* because high-school students (and even some high-school teachers) might not be able to understand research knowledge. However, Brousseau (2006) describes didactic transposition as *regrettable,* in that simplifying knowledge runs the risk of oversimplification and misinterpretation of the knowledge.

Nevertheless, the translation of knowledge is linked to the concept of *institutional relativity of knowledge.* In the context of this study, *institution* refers to a high school. To illustrate this, Christiansen and Rump (2008) described how the concept of thermodynamics is explained differently by the chemical engineering, mechanical engineering and physics departments respectively at the Technical University of Denmark. Likewise, trigonometry at different schools is taught differently. Among other reasons this might be because of teachers' expertise and experience, or the time required to complete the curriculum. In this regard, Chevallard and Bosch (2013, p.3) explained that there "is no such thing as an eternal, context-free notion or technique, the matter taught being always shaped by institutional forces that may vary from place to place and time to time". Bosch and Gascón (2006) argued that these different interpretations of knowledge are not necessarily correct or incorrect, but respond to specific institutional requirements. The same can be said about why teachers instruct in a very procedural way. For example, they might feel pressurised to complete certain topics because of high-stakes examinations, in which case the 'institutional requirement' would be to complete a specific number of topics before students are tested on those topics. In fact, when Stigler and Hiebert (1999) compared (mathematics) teaching styles between the United States and Japan, they found that the motto of American teachers was "learning terms and practising procedures" (p.27); in contrast to Japanese teachers, who taught for conceptual understanding. Additionally, Japanese teachers participated in long-term, ongoing, incremental and continuous improvement of their teaching.

ATD also has broader applications. Rodríguez et al. (2008) tried to reformulate metacognition in problem solving using ATD, by using ATD to overcome cultural differences between "mastering a curriculum" and "implementing it in new situations" (p.290). In this regard, ATD postulates that institutional conditions and restrictions should be investigated to explain why certain mathematical activities are difficult or easy to implement. In general, but specifically with simplifying trigonometric expressions and proving trigonometric identities, we can consider a praxeology with either a praxis or a logos. These instances occur when students can solve a question but do not know *why*, or they can describe a question but not know *how* to solve it.

Rodríguez et al. (2008) realised that to define the construct of 'metacognition' in terms of ATD's praxeologies, they had to accept that metacognition belongs to two different theoretical approaches. One is psychology. The other theoretical approach is in mathematics education, where 'metacognition' is framed as "classical problem solving" (p.299). Rodríguez et al. (2008)

contended that a research problem in one theoretical approach is not necessarily meaningful in a different approach. However, they argued that metacognition in mathematics education is based on what they termed “Pólya’s problem” (p.288) – that is, the methodologies employed by teachers and students to solve non-routine problems. Rodríguez et al. (2008) stated that Pólya’s problem can be reformulated into mathematical praxeologies. Moreover, the new framing of Pólya’s problem in terms of the ATD can be related to a new teaching proposal within ATD called *Study and Research Courses.* These entail a pattern based on questions and answers. Chevallard (2007) contended that these questions and answers are the essence of how knowledge is gained, and thus are the essence of a praxeology. For example, Chevallard (2007) contended that a question might be raised in an institution (in the case of this study, the question relates to the simplification of trigonometric expressions and the proving of trigonometric identities), and the answer to that question has the structure of a praxeology.

Chevallard (2004; 2006) introduced Study and Research Courses (SRC) as a step-by-step process to generate an *answer to* a *question.* In the case of the networking investigation of Rodríguez et al. (2008), the *question* was how to formulate ‘metacognition’ within ATD. Next, what Chevallard calls a ‘study community’ must be formed to investigate the problem. In this regard, Rodríguez et al. (2008) tasked two groups of Spanish grade 11 students with investigating comparative tariffs for mobile telephone services. Both groups revealed the complexity that telephone companies incorporate when calculating a customer’s monthly account. Their results motivated the students to write to their Minister of Industry to complain about this “tariff complexity” (p.295) that phone companies employ to disorientate customers and make it difficult to compare phone tariffs. It is precisely these results that this study wished to emulate, in tasking students with answering questions of simplification of trigonometric expressions and the proving of trigonometric identities. That is, this study wanted to show that students can produce unique and thought-provoking solutions through productive struggle.

Rodríguez et al. (2008) concluded that problems that are associated with ‘metacognition’ – such as observing, preparation, testing, self-regulation, etc. – could also be described in terms of institutional (referring to the school or organisation), didactic and mathematical praxeologies.

Regarding this study, the participating school district requires the teacher to complete certain topics in trigonometry (such as the simplification of trigonometric expressions and the proving of trigonometric identities). For this study, the students were only introduced to $sin^2x + cos^2x = 1$ and $tanx = \frac{sinx}{cosx}$. This restriction was reflected in the students' solutions to the activity questions. Furthermore, in their exploration of context-based learning systems based on generic scenarios, ATD helped Tetchueng et al. (2008) to structure and refine the scenario model and the didactical setting. The researchers' tool of scenarios was used to describe the learning and tutoring events required to acquire a certain knowledge domain (for instance, chemistry) and the expertise to solve a problem. This scenario structure can be recycled in the sciences, for instance in botany and biology. A similar approach was possible with the simplification of trigonometric expressions and the proving of trigonometric identities. That is, at the beginning of the study, the participating teacher provided a 'scenario' of how to use trigonometric identities in the simplification of trigonometric expressions.

2.2.1.1 ATD in mathematics education

One of the basic tenets of ATD is *didactic transposition*. Didactic transposition refers to the deconstruction and reconstruction of knowledge to make it easier to teach. This means the transformation of knowledge that should be taught – i.e. the knowledge that is developed by researchers, or prescribed by state or provincial authorities – is transformed into the knowledge that is taught by the mathematics teachers, then into the knowledge that is learnt by the student. Thus, it is essential to understand didactic transposition to comprehend the interaction between student, teacher and mathematical knowledge (Bosch & Gascón, 2006; Chevallard, 1985).

Another tenet of ATD is that of *human activity,* and particularly *mathematical activity,* which can be modelled by praxeologies. A mathematical praxeology will describe the mathematical activity from the point of view of the student, and a didactic praxeology from that of the teacher. The mathematical praxeology from the student's point of view was discussed in section 2.2.1. For the didactic praxeology, in the practical block (the praxis or 'know how') from the teacher's perspective the *task type* refers to the questions issued by him or her that relate to the simplification of trigonometric expressions and the proving of trigonometric identities; the *technique* refers to the method or procedure used by the teacher to illustrate the simplification of trigonometric expressions and the proving of trigonometric identities. For example, the teacher might use trigonometric identities to illustrate the simplification of a trigonometric expression. The technology is to simplify the trigonometric expression or show that the left-hand side and right-hand side of a trigonometric identity are equivalent. The theory which justifies the technology is the application of trigonometric identities in conjunction with prior learnt algebraic skills.

In the theoretical block (the logos or 'know why') for both student and teacher instances, the t*echnology* justifies the technique used to solve a question, and the *theory* justifies the technology.

Larger praxeologies can be built up from smaller, less intricate praxeologies. Chevallard (1999) used the scale that follows to explain the hierarchy of praxeologies:

Specific praxeology: Here a single type of task is given, a specific technique is applied, and the technology is implied. For example, 'Simplify the trigonometric expression: $5 + sin^2x + cos^2x$.' The technique to be employed is to replace $sin^2x + cos^2x$ with 1, then add five.

The technology – that is, the justification of the technique used to simplify the trigonometric expression, which is 'trigonometric identities' – is implied.

Local praxeology: the integration of specific praxeologies connected by a common technology. For example, 'Simplify the trigonometric expression: $sinxcos^2x - sinx$.' Here, the common technology is the integration of factorisation and trigonometric identities.

Regional praxeology: the integration of local praxeologies with a common mathematical theory. For example: to simplify the trigonometric expression $sin^2x + tan^2xsin^2x$, the common theory is the integration of algebra (factorisation) and trigonometry (the identity $sin^2x + cos^2x = 1$) in simplifying trigonometric expressions.

Global praxeology: The integration of different regional praxeologies. For example, to prove that $3tanx - \frac{1}{tanx} = \frac{3-4cos^2x}{sinxcosx}$, regional praxeologies include (if starting with the left-hand side of the equation) the manipulation of algebraic fractions and the manipulation of the trigonometric expression $sin^2x + cos^2x = 1$. However, if starting with the right-hand side of the equation, instead of replacing $sin^2x + cos^2x$ with 1, the reverse should be done, that is 3 should be replaced by $3sin^2x + 3cos^2x$. This strategy in conjunction with algebraic manipulation of fractions would constitute the integration of regional praxeologies.

Furthermore, Garcia et al. (2006) explained how to construct a praxeology from a didactic perspective in an educational setting. The praxeology presents itself via six *moments,* which are different approaches for learner activities. The first moment is that of the *first encounter.* This happens when the student first observes a question associated with the praxeology. For example, a student might be asked to simplify $tan^2x - tan^2xcos^2x$. The second moment is that of *exploration.* This happens when the student finds different techniques to solve the question observed in the first encounter. From the previous example, the student might replace tan^2x with $\frac{sin^2x}{cos^2x}$ and try to subtract the algebraic fractions. The third moment is the *technical work* moment, which relates to using and improving techniques used for the first encounter. During this moment, if a student replaced tan^2x with $\frac{sin^2x}{cos^2x}$, and subtracted the two algebraic fractions, his or her expression would have been $\frac{sin^2x - sin^2xcos^2x}{cos^2x}$. At this point, a different strategy (factorisation) is needed. The fourth moment is the *technological theoretical,* in which alternative techniques are assessed and a technological discourse takes place. For the fourth moment, to simplify $tan^2x - tan^2xcos^2x$, a student might realise that to collect a common factor – namely $tan^2x(1 - cos^2x)$ – would be a better option. The fifth moment is that of *institutionalisation,* when a student tries to discern and identify the praxeology. For the example under discussion, the student connects trigonometric identities with algebraic manipulations. In this moment, $1 - cos^2x$ could be replaced with sin^2x, and tan^2x with $\frac{sin^2x}{cos^2x}$. The last moment is that of *evaluation,* which aims to assess the value of the constructed praxeology. The 'reworked' $tan^2x - tan^2xcos^2x$ would yield $\frac{sin^4x}{cos^2x}$. A student could evaluate the praxeology by comparing the 'reworked' trigonometric expression with the original one to see if it had been 'simplified'.

ATD provides a detailed model of the *levels of didactic codetermination* by which the observer may identify the constraints or conditions that stifle or promote a praxeology (see Table 2.1 below). According to Chevallard (2002b), these levels of codetermination are: subject, theme, sector, domain, discipline, pedagogy, school, society, and civilisation. In the instance of this study, each level of didactic codetermination should be interpreted as follows. First, subject: this refers to the task of simplifying trigonometric expressions and proving trigonometric identities at high-school grade 11 level. Second, theme: this refers to the section in which the subject resides. Simplification of trigonometric expressions falls under *fundamental identities and* proving trigonometric identities under *proving identities.* Third, sector: this refers to the section in which the theme resides. Both fundamental identities and proving identities fall under *analytic trigonometry.* Fourth, domain: this refers to the section where the sector resides. Analytic trigonometry falls under *trigonometry;* which is part of the fifth level of codetermination, namely the discipline of *mathematics.* Sixth, pedagogy: this refers to the underlying teaching methodologies used. In this study, both procedural (traditional) and conceptual (productive struggle) techniques were used to investigate the simplification of trigonometric expressions and the proving of trigonometric identities. Seventh, school: the school where the study was to take place, which was AWS High School, described previously. Eighth, society: this refers to the organisation that makes decisions regarding the school, which is the state department of education. And ninth, civilisation: this is the civilisation in which the study took place, which is the American culture.

Table 2.1: Summary of the levels of didactic codetermination

Level	**Praxeology**
Civilisation	American culture
Society	State Department of Education
School	AWS High School
Pedagogy	Procedural and conceptual teaching techniques
Discipline	Mathematics
Domain	Trigonometry
Sector	Analytic trigonometry
Theme	Fundamental identities and proving identities
Subject	Simplifying trigonometric expressions and proving trigonometric identities.

The third tenet of ATD is the investigations of an *ecology* of mathematical and didactic praxeologies. An ecology consists of *conditions* that advance or promote a praxeology, and *constraints* that impede the praxeology. The ecology is determined by the *levels of didactic* *codetermination* by which the observer can identify the constraints or conditions that stifle or promote a praxeology.

For example, take the ‘ecosystem’ of *school, pedagogy, discipline and domain.* Issues that occur in trigonometry (the domain) might be attributable to the school – perhaps if the school has a culture of not offering regular courses in trigonometry; or an issue might arise in the pedagogy, such as teaching styles used that hinder deep learning.

In their investigation into didactic constraints on teacher practice, Barbé et al. (2005) used Chevallard’s (2002) version of ATD, which proposes a hierarchy of levels of codetermination of mathematics praxeologies. Each of the different mathematical praxeologies (punctual, local, regional) introduces different restrictions on the teacher, illustrating the interplay between mathematical and didactic praxeologies.

Winsløw et al. (2014) used ATD to explain the factors that influence an institutional approach to *University Mathematics Education (*UME). Using ATD, they investigated the challenges that university teachers and students at a Belgian university face when teaching *duality* in linear algebra; the concept of *limits* in an introductory calculus course at a Canadian university; and *mathematical modelling* for first-year natural science students at a Spanish university. ATD explained the problem that students have transitioning from secondary school to university. At secondary school, the emphasis is on the practical block (*task type* and *technique*); and the theoretical block typically resides with the teacher. But at most universities the theoretical block is introduced first, with less emphasis on the practical block. In this study, ATD was applied to high-school students’ productive struggles in trigonometry. The practical block consisted of techniques that the students developed to simplify trigonometric expressions and prove trigonometric identities. The theoretical block provided justification to the teacher and the researcher for using certain techniques to simplify trigonometric expressions and prove trigonometric identities.

One of the aims of this study was to identify what part of the theoretical block influences the practical block – what type of teacher questioning and noticing influences students to apply certain *techniques* in solving which *types of activities.* Thus, just as university mathematical praxeologies must be adapted to students' secondary-school praxeologies to accommodate the transition between secondary school and university, teacher questioning and noticing should accommodate student 'transitioning' to find solutions to their productive struggles. Winsløw et al. (2014) reported that to make the transition to the duality in linear algebra more tangible for first-year mathematics students, university educators tried to find aspects of students' secondary-school praxeologies that could be generalised to linear algebra. Likewise, when the participating teacher in this study noticed students struggling, he generated questions to which the students knew the answers; they could then try to relate their prior knowledge to their current struggle. For example, when students struggled with simplifying expressions such as $\frac{7}{sinx} + \frac{8}{cosx}$, the teacher could relate such an example to $\frac{7}{9} + \frac{8}{11}$ and explain the similarities between the two expressions.

Just as Putra and Witri (2017) used didactical and mathematical praxeologies from the ATD as a framework to study in-service elementary school teachers' knowledge of fractions in Indonesian schools, this study used didactical praxeologies (although not as a standalone research question) to analyse teacher noticing and questioning in promoting or stifling productive struggles when simplifying trigonometric expressions and proving trigonometric identities. Of interest are the technologies the teacher employed to guide the students in using certain techniques to solve a question.

This section showed how ATD is used in mathematics education, and how it was applied in this study. The next section will explain the relevance of ATD to this study more broadly.

2.2.1.2 ATD relevance to this study

This study seeks to investigate the nature of the productive struggles students experience when simplifying trigonometric expressions and proving trigonometric identities. It then compares the struggles that students experience when simplifying trigonometric expressions to those experienced when proving trigonometric identities. In addition, using the NEA this study sought to examine what types of productive struggles from the learning activities remained unresolved during the assessment on the simplification of trigonometric expressions and proving of trigonometric identities.

Chevallard (1992) used the two main components of a praxeology, the practical block and the theoretical block, to propose two aspects of a human mathematical activity.
The practical block consists of two subcomponents: *type of activity* and *technique*. *Type of activity* concerns itself with *what* the student must do, for example simplifying a trigonometric expression or proving trigonometric identities. *Technique* deals with *how* to solve the question – that is, what technique the student will employ to simplify a trigonometric expression. However, simplifying a trigonometric expression is not that straightforward. Before starting to simplify a trigonometric expression, the student should have a strong sense of what 'simpler' means – a notion with which Delice (2002) took issue when she studied the understanding of and performance in simplifying of trigonometric expressions by English and Turkish students. In the context of this study, simplifying a trigonometric expression means making the expression easier than how it was originally presented. That is, the countable terms in the resultant trigonometric expression will be fewer than in the original trigonometric expression. For example, the expression $tan^3xsinxccos^3x$ consists of three terms, namely $tan^3x, sinx$ and cos^3x. However, when simplified, the resultant expression has been reduced to one term, sin^4.

Likewise, proving trigonometric identities means illustrating the equivalence of two trigonometric identities. Proving trigonometric identities can become complicated if the student does not know which side of the equation to start with. But whatever method chosen to address a mathematical question, the anthropological approach requires a technique to solve it.

Similarly, the theoretical block has two subcomponents: technology and *theory*. *Technology* deals with justifying the *technique* used by the students, and *theory* is used to explain the *technology*. Thus, *theory* has the same relationship to *technology* as *technology* has to *technique*. For example, when students try to prove $\frac{1}{3}tanx + \frac{2}{5tanx} = \frac{5sin^2x+6cos^2x}{15sinxcosx}$ they could start on either side of the equation. If they start with the left-hand side, a technique that they could employ is to add the two (trigonometric) fractions $\frac{1}{3}tanx + \frac{2}{5tanx}$ together by finding a common denominator and using algebraic processes in collaboration with trigonometric identities to show that the solution is the same as the right-hand side. The technology that justifies their technique is the simplification of trigonometric expressions. The theory underlying the technology is the proving of equivalent expressions.

These four components – *types of activity, technique, technology* and *theory* – are termed a mathematical praxeological organisation. A mathematical praxeology based on a distinct type of problem in a given establishment is called a *punctual mathematics praxeology.* Expanding this notion, a *local mathematical praxeology* is a set of punctual *mathematics praxeologies* that can be explained by a common set of technologies. From the viewpoint of this study, the simplification of trigonometric expressions and the proving of trigonometric identities are two punctual mathematical praxeologies (the simplification of trigonometric expressions is based on 'simplification' and the proving of identities on 'proving' technologies) guided by trigonometric identities, and can thus be viewed as a *local mathematical praxeology.*

In his investigation of the ATD perspective on teaching in Indonesia, Putra (2017) concluded that didactic knowledge is closely linked to a teacher's mathematical knowledge. Likewise, in this study, teacher knowledge of trigonometric identities and different methods of proving identities and simplifying trigonometric expressions is important, since guidance from the teacher directly influences the process of supporting students' productive struggle. However, only two trigonometric identities were used.

Also, when Barbé et al. (2005) investigated didactic restrictions on teacher practice when dealing with the limits of a function, they found two types of didactic restriction: specific and generic, which deal with the nature of the knowledge to be taught and how to teach the mathematical topic respectively. Regarding this study, the topic under investigation, trigonometry, formed part of the regular syllabus. Nonetheless, the researcher decided to use only the identities $sin^2x + cos^2x = 1$ and $tanx = \frac{sinx}{cosx}$. These 'restrictions' made for easy comparison between the questions of the activities. Next, the researcher recommended minimal instruction from the teacher, since the investigation was to concern itself with students' productive struggles.

2.2.2 The Developing Cognitive Abilities Test (DCAT)

The Developing Cognitive Abilities Test (DCAT) is a group administered test that was developed by Beggs & Mouw (1980). The DCAT reflects Bloom's (1956) three stages of cognitive development. Under the DCAT, the basic cognitive abilities level reflects Bloom's knowledge and comprehension level. The application abilities level under DCAT reflects Bloom's application level and the critical thinking abilities under DCAT reflects Bloom's analysis and synthesis level.

The exercises in the activities were sequenced using the DCAT. This means that the exercises were organised in three groups of increasing complexity, i.e., easy, medium, and difficult. The easy exercises related to the DCAT's *Basic Cognitive Abilities,* referred to as DCAT 1; the medium exercises related to *Application Abilities,* referred to as DCAT 2; and the difficult exercises related to *Critical Thinking Abilities,* referred to as DCAT 3.

2.2.3 How students learn

This section clarifies the method used to explain how students learn mathematics. Students may go through productive struggles to accomplish a mathematical task; however, they must be mathematically proficient, which means that they should have conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning.

2.2.3.1 Productive struggles

A student's 'productive struggle' is the student's "effort to make sense of mathematics, to figure something out that is not immediately apparent" (Hiebert & Grouws, 2007, p.287). However, Hiebert and Grouws (2007) cautioned, struggling that engages students in exploring a problem is not a once-off phenomenon, but is continuously evolving. Schoenfeld (1988) argued that this type of struggle takes time and effort. In support of the concept of struggle, Hiebert and Wearne (1993) asserted that "all students need to struggle with challenging problems if they are to learn mathematics deeply" (p.6). Kapur and Bielaczye (2012) agreed, arguing that for students "to developing deeper understandings" (p.46) means engaging with a mathematical problem using components of perseverance, potential disappointment and struggle. The disappointment usually comes when the student does not solve a question successfully. However, Kapur (2010; 2011) maintained that the act of struggling, even without success, may have long-term benefits. Kapur (2009) also argued that these unsuccessful struggles may assist with students' long-term remembering; and Hiebert and Wearne (2003) contended that unsuccessful struggles lead to deep learning. But student struggles with and dislike of mathematics are well documented (Acharya, 2017; Feldman et al., 2014; Gafoor & Kurukkan, 2015). Struggle in mathematics is seen by educational stakeholders as a 'problem', a learning difficulty that they attempt to eliminate through diagnosis and remediation (Adams & Ham, 2008; Borasi, 1996).

Unlike disciplines that require rigid thinking and robotic following of procedures, mathematics is exactly the opposite. It is a subject that is constantly changing; new theorems are being discovered and must be proved, while old ones may be expanded and applied in real-life situations after decades of lying dormant. Nonetheless, "the study of mathematics is apt to commence in disappointment" Whitehead (2017, p.7). It stands to reason that the way students learn mathematics should also evolve. During this study, the participating teacher allowed students to collaborate, in pairs, before asking them to demonstrate the solutions to the questions on the blackboard. The students enjoyed being paired with classmates because they could 'bounce ideas off each other'. Furthermore, the arrangement resulted in a relaxed atmosphere, in which seeking help was not seen in a negative light, but as an opportunity to help a fellow classmate. This is aligned with Carter (2008) and Kapur (2011), who posited that a 'wrong' reply need not be seen as a failure, but as an opportunity to learn from someone else – where students in a risk-free environment use each other's ideas as learning resources, and to support and encourage a fellow student to persevere in his or her problem solving. This idea of 'stress-free learning' sat well with Ambrose et al. (2010), who concluded that relaxed students show affective responses in the most difficult classes when fear as an educational barrier is eradicated.

Other researchers – for example, Ambrose et al. (2010) and Baldwin et al. (1997) – agree that there is a correlation between a student's performance in class and the student's enjoyment of the class. Importantly, students love the idea of doing 'homework' in class (Bempechat et al., 2011; Cox & Singer, 2011). In addition, Else-Quest et al. (2010) reported that students are more likely to take additional mathematics modules if they have confidence in their mathematical abilities, stating also that these modules lean towards a higher level than those taken by their less confident peers. Conversely, if students' expectations of their own success are not met, one sees a decline in their confidence (Ahmed et al., 2013).

The American College Testing (ACT) programme's *Condition of College and Career Readiness* report of 2016 shows that 25% of South Carolina students who sat for the examination tested career ready, compared to a national score of 41%. Furthermore, the 'difficult' questions on the ACT examination are not from the culmination of the Advanced Calculus syllabus; rather they constitute insightful questioning of the student's prior knowledge of basic mathematics. It stands to reason that mathematics should be taught in such a way that students develop a deep understanding of any concept that is presented to them. In the preceding paragraphs, this study laid out a justification for why supporting productive struggle would be an ideal methodology for students to gain deeper understanding when learning mathematics.

In addition, the *South Carolina College- and Career-Ready Standards for Mathematics* (SCCCR) resource states that a mathematically literate student should "use critical thinking skills to justify mathematical reasoning and critique the reasoning of others", and "use a variety of mathematical tools effectively and strategically" (p.138). Moreover, the South Carolina Educational Oversight Committee reported that South Carolina ranks near the bottom of all states for the 2020 ACT results. The Committee stated that "there is no easy solution to these data, but we have to do better than rinse, repeat, and hope for better results".

Given this scenario, why are South Carolina's results so poor? It is precisely this dilemma that this study wishes to illuminate and try to address through some innovative learning strategies involving the exploration of students' productive struggles. There was a huge discrepancy between the participating school's advanced mathematics class average grade and their score on the mathematics portion of the ACT. What motivated this study was the need to investigate how students could use productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities to improve the learning and teaching of trigonometry at high-school level.

This study concurs with the findings of Porter et al. (1988), which allude to the fact that teachers focus their attention on developing computational skills. The practice of teaching mechanical procedures is widespread in the schooling system, even in "good schools", as Schoenfeld (1988) reported. Likewise, Boston and Wilhelm (2017), as well as Stigler and Hiebert (1999), affirmed that under conventional teaching, students are expected to memorise known procedures and timesaving methods to solve non-challenging questions. Conventional teaching refers to direct instruction by the teacher and the assessing of that knowledge.

Typically, little teacher noticing (and consequently, little challenging questioning) takes place to promote classroom participation. Instead, this type of discussion among students is seen as a disturbance in the classroom that is not conducive to the 'classroom atmosphere'. Similarly, Staples (2007) agreed that in conventional classrooms, mathematical knowledge is typically not discovered, but rather acquired from textbooks, and most importantly from the teacher. However, Freeman et al. (2014) challenged this notion of where knowledge authority lies, reporting that active learning increases assessment outcomes by less than half a standard deviation (SD), and that lecturing increases the failure rate by 55%. The results obtained by Freeman et al. (2014) hold for different class sizes, course types, and course levels.

Investigating the struggles of young teenage students in the mathematics classroom, Warshauer (2011; 2014; 2015) identified four main types of productive struggle: *getting started, carrying out a process, experiencing uncertainty in explaining and sense making*, and *expressing misconceptions and errors.* Table 2.2 below describes these four categories of productive student struggles.

Table 2.2: Types of student productive struggles from Warshauer (2011)

Type of struggle	Description
Getting started	When confronted by a mathematical question such as simplifying a trigonometric expression, students may utter expressions of frustration and confusion. These emotional outbursts are not accompanied by any evidence of work. Students may claim that they have never seen this type of mathematical question before, or have forgotten how to do it – hence, students do not know how to start the process of solving the task.
Carrying out a process	When students are confronted by a challenging mathematical question, such as proving a trigonometric identity, they may reach an impasse. This may be linked to their inability to carry out an algorithm such as taking out a common factor during the factorisation process. Also, students may find it difficult to follow a known procedure, such as adding like (trigonometric) terms. Students may also forget facts or formulae required to solve the task.
Experiencing uncertainty in explaining and sense-making	When working in small groups, students may find it difficult to explain their work to their group or to the entire class when asked to by the teacher. A student may declare that he or she knows what they have produced is the correct answer, without giving any evidence.
Expressing misconceptions and errors	Errors can be classified into errors due to carelessness, and conceptual errors. Conceptual errors occur when students fail to observe the correct relational ideas when solving a question. A misconception is usually not wrong thinking; however, it can be interpreted as an indication of deep-seated misplaced ideas that are used to justify the process of finding a solution to a question. These may manifest as local generalisations.

To evaluate the teacher's responses to the students' productive struggles, this study used categories suggested by Warshauer (2015). These categories included *telling* the students the correct answer; giving the students *directed guidance*, where the teacher guides students based on the teacher's thinking; *probing guidance*, where the guidance is based on student thinking and student self-reflecting; *affordance*, where the teacher asks the students to justify their contribution to a solution, and guidance from the teacher is based on student thinking; and lastly, *unfocused*, or *vague*, where the teacher does not direct the students to a strategy based on their thinking, but rather suggests a broad strategy that is not helpful in solving the question. Table 2.3 below categorises teacher responses to productive (or unproductive) struggles (Warshauer, 2015). In the table, T = teacher and S = student**.** The reader is advised that some examples are from the students' activities.

Table 2.3: Teacher response to productive (or unproductive) struggles

Response	Description	Consequence	Example
Telling	Entails supplying students with information.	Removes the struggle and allows the students to make progress using a prescribed strategy.	S: I am struggling to simplify $cos^3x + cosxsin^2x$ T: Factorise out a common factor. (Unproductive struggle)
Directed Guidance	Entails redirecting students to a strategy that is aligned with the teacher's thinking.	Stifles original work from the students.	T: Does your answer look 'easier' or more difficult than the question? S: More difficult. T: Then we need to try a different strategy. (Productive struggle)
Probing Guidance	Entails determining what a student is thinking, promoting student self-reflection, and offering suggestions based on student thinking.	Encourages thinking by the students	Simplify: $cosxsin^2x - cos$ S: cos$x \cdot (sin^2x - cosx)$ T: That's not a bad idea… it's not exactly what you do, but it's not a bad idea. (Productive struggle)
Affordance	Entails asking students to convey what they have done. Encourages their way of thinking, with minimal intervention.		Simplify: $sin^2x + cos^2x + 5$ S: I thought about changing $sin^2x + cos^2x$ with 1 T: Keep going… (Productive struggle)

Unfocused or vague	Does not direct students to a certain strategy; neither does it build on student thinking. The teacher provides a suggestion that is too broad to be helpful.	Redirect students' focus away from a possible solution	S: *[writes* $\frac{cos^2x+sin^2xcos^2x+sin^4x}{sin^2x}$ *on the whiteboard]* T: Where do we go from there? S: *[silence]…*cancel? T: Where do we go from there?! (Unproductive struggle)

2.2.3.1.1 How does productive struggle fit into the ATD?

Given any mathematical task – in the case of this study, to simplify a trigonometric expression or to prove a trigonometric identity – a student may experience difficulties in solving the problem. As shown above, these difficulties might be in *getting started, carrying out a process, uncertainty in explaining and sense making,* or *expressing misconceptions and errors.* The teacher response may vary from *telling the student the appropriate technique,* to *giving the student directed guidance, probing guidance or affordance.* The underlying theory depends on the technology the teacher employed to elicit the best technique to solve a mathematical task. Next, this study explains productive struggles, both from the students' and from the teacher's viewpoint.

From the students' perspective, 'task type' refers to a question related to the simplification of trigonometric expressions or the proving of trigonometric identities. From the teacher's viewpoint, his or her task type would be different ways to support the students' productive struggles when confronted by a question relating to simplifying trigonometric expressions or proving trigonometric identities. Following this the students may employ certain techniques to solve their task type. These techniques are related to the productive struggle that the students experience, and may include replacing trigonometric expressions with trigonometric identities, applying algebraic processes in trigonometry, and so forth. For example, getting started may result in (incorrectly) factoring. The teacher may employ one of telling, directed guidance, probing guidance, affordance, or give unfocused or vague guidance as a technique to support his or her task type. The technology that underscores the students' technique is insufficient prior knowledge, or the misinterpretation thereof. The technology that underscores the teacher's technique is methods of teaching and the transfer of knowledge. The theory that underscores the students' technology is the advantages of productive struggles, while the theory that supports the teacher's technology is how to (productively) support different learning styles.

2.2.3.2 Mathematical proficiency

In this section this study sought to explore how mathematical proficiency (or a lack thereof) influences students productive struggles when simplifying trigonometric expressions and proving trigonometric identities. In particular, the definition of mathematical proficiency put forward by Kilpatrick et al. (2001), described subsequently, was used to investigate the relationship between the students' productive struggles and their mathematical proficiency.

During the early 20th century, success in mathematics learning corresponded to changes in society and schooling systems (Kilpatrick et al., 2001). The first half of the twentieth century saw the mathematics education movement emphasising computational procedures for arithmetic. In the following decade the emphasis broadened to include understanding the structure of mathematics and its unifying ideas, rather than purely computational skill. Ironically, the next era exemplified the 'back to basics' movement, in which the focus was on quick and accurate computation. Lastly the 'reform movement' emphasised problem-solving skills, coupled with communicating mathematics to others meaningfully and connecting mathematical ideas.
Consequently, Kilpatrick et al. (2001) defined mathematics proficiency in terms of five interwoven strands: *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning,* and *productive disposition.* Each strand will be explored in the subsections to come.

Kilpatrick et al. (2001) stated that these strands are interdependent and represent different parts of a multi-part mathematical unit, which was confirmed by Schoenfeld's (2007) assertion that mathematics proficiency is multidimensional. On the other hand, in conjunction with *didactic transposition* Schoenfeld (2007) placed proficiency at the centre of what he calls "cognitive revolution" (p.59), which he defined as the move from emphasising knowledge as a construct of what one *knows* to a construct of what one *knows and* can *use* in *different domains.* In the context of this study, a student might *know* some trigonometric identities, but have difficulty applying that knowledge in new circumstances. For example, a student might know that $tanx = \frac{sinx}{cosx}$, and yet fail to simplify $tanxcosx$.

Schoenfeld (2007) warned that individual skills are a necessity for doing mathematics, but that a student knowing each skill by itself does not mean he or she is mathematically proficient. According to Schoenfeld (2007), mathematicians in academia or industry (and thus, presumably, mathematically proficient people) should have the following characteristics: firstly, they should be *good problem solvers* who are malleable and inventive in their thought; secondly, they should be able to apply *existing mathematical* knowledge in *new contexts* and to *new problems;* thirdly, and consequently, they are able to use *existing* mathematical results to find *new results;* fourthly, they can *approach* a question from *different angles;* fifthly, they *accept hard mathematics problems* on the assumption that they will be able solve those problems; and lastly, mathematically proficient people are *tenacious.* From a didactic perspective, moreover, Schoenfeld asserts that these traits can be taught at school, and they are applicable when students are engaged with the simplification of trigonometric expressions and the proving of trigonometric identities.

2.2.3.2.1 Conceptual understanding

Kilpatrick et al. (2001) refer to *conceptual understanding* being when students have coherent and useful understanding of mathematics; this study uses their definition. Students with conceptual understanding arrange knowledge in units, and this enables them to acquire new knowledge, by connecting new ideas with existing ones (Hiebert & Carpenter, 1992). The process of struggling may help students to learn with understanding (Hiebert & Wearne, 1993), and thus promote acquisition of new knowledge, and the establishment of connections between what they already know and new knowledge – leading to knowledge retention. Furthermore, items of information acquired with understanding are related to one another, and thus easier to remember and reconstruct when forgotten (Hiebert et al., 1996).

Often educators try to establish evidence of conceptual understanding by asking students to explain received information verbally. However, Geary (1995) warned that students often understand knowledge *before* they can verbalise any given information. Conceptual understanding produces students who can see connections between related concepts. For example, an ability to see the resemblance between $\frac{1}{5}-\frac{1}{7}$ and $\frac{1}{x}-\frac{1}{y}$ demands *conceptual* understanding of the mechanics of fractions. Expanding this idea to $\frac{1}{sinx}-\frac{1}{cosx}$, demands even greater understanding, to see the relation between algebraic and trigonometric fractions. In the context of this study, conceptual understanding between numerical, algebraic and trigonometric fractions is important.

Moreover, Hiebert and Wearne (1986) posited that although rote learning techniques such as mnemonics make it easier to perform mathematical operations, they do not lead to understanding. In trigonometry, the mnemonic SOHCAHTOA, (pronounced ‘soak-a-toe-ah’) is easily remembered by students. This is a shortcut reminder that ***s**ine* is equal to ***o**pposite* divided by *hypotenuse*, ***c**osine* is equal to ***a**djacent* divided by ***h**ypotenuse*, etc. However, this does not guarantee that a student will know how to identify an adjacent or opposite side for a specific angle.

2.2.3.2.2 Procedural fluency

Procedural fluency refers to the effective, efficient, malleable, accurate and skilful use of procedures in appropriate situations (Kilpatrick et al., 2001). A theme that runs through this study is the strong connection between procedural fluency and conceptual understanding. The assumption here is that students in grade 11 have attained a level of procedural fluency that allows them to perform operations on two numeric and algebraic rational fractions.

Algebraic rational expressions and trigonometric rational expressions, on the other hand, are both fractional in nature, but significantly more must be done to calculate a denominator. For example, when adding the fractions $\frac{1}{x^2-2x-3}+\frac{1}{x^2+6x+8}$, the student should first realise that both denominators are quadratics that must be factored before finding a common denominator; but $\frac{5}{cosx}-\frac{2}{cosx}$ can be considered a procedural question, as the fractions have a common denominator.

Most mathematics teachers concentrate their attentions on developing computational skills (Porter et al., 1988), as was evident from speaking to teachers at the participating school. Also, Chua (2017) argued that procedural teaching of factorisation and expansion deprives students of the opportunity to discover or generate new mathematical knowledge. Didis et al. (2011) pointed out that even students who know how to factorise quadratics do it procedurally, without conceptual understanding. Furthermore, Tirosh et al. (1998) claimed that procedurally-taught students find it difficult to produce answers that ‘lack closure’. They tend to think that an open expression is incomplete, and that they must find the answer for it.

However, procedural and conceptual understanding go hand in hand. Long (2005) focused on conceptual understanding when she instructed pre-service teachers on the use of number bases. Zuya (2017) and Khashan (2014) found that prospective mathematics teachers perform better on procedural knowledge than on conceptual knowledge. But conventional teaching engages students in procedural rather than conceptual activities, with limited opportunities to engage in mathematical reasoning. Students are drilled to master known procedures for solving questions that are based on those procedures (Boston & Wilhelm, 2017; Stigler & Hiebert, 1999).

Hiebert and Lefevre (1986) identified two unique kinds of knowledge: conceptual knowledge and procedural knowledge. Similarly, Skemp (1976) defined relational and instrumental understanding as being equivalent terms. According to Skemp, relational understanding is the capacity to deduce specific rules and processes from more general mathematical relations. Instrumental understanding is the capacity to apply rules to solve a question without insight. For example, to simplify $\frac{sin^2x+2sinxcosx+cos^2x}{sinx+cosx}$, a student with instrumental understanding might replace $sin^2x + cos^2x$ with 1 and yield $\frac{1+2sinxcosx}{sinx+cosx}$ as their solution. And though there appears to be a difference between procedural and conceptual understanding in the instruction of mathematics, Wu (1999) argued that the concepts are intertwined. Furthermore, Schollar (2004) argued that there is evidence that poor understanding on the part of teachers regarding the constructivist approach has led to learners having neither conceptual nor procedural knowledge.

2.2.3.2.3 Strategic competence

Strategic competence refers to a student's ability to articulate a mathematical question, represent the question, and resolve it. Kilpatrick et al. (2001) referred to this area in mathematics education as 'problem solving and problem formulation'. Strategic competence involves the avoidance of a phenomenon called 'number grabbing' by Littlefield and Rieser (1993). Number grabbing occurs when students select numbers and perform arithmetic operations on them without understanding the problem. In particular, Littlefield and Rieser (1993) investigated the influence of extraneous information (which they call 'information discrimination') on word problems. They found that less successful mathematics students were not as flexible as others in the allocation of solution time for testing questions. Although Kilpatrick et al. (2001) defined flexibility in terms of students' ability to discern which strategies to use when solving questions, both theories require students to spend time on a question. The participating teacher in this study referred to the time that students spend struggling as being 'uncomfortable' for him. This discomfort arises from wanting to allow students ample time to struggle with simplifying trigonometric expressions and proving trigonometric identities, without interfering with their struggles.

Wilson and Heid (2011) posited that strategic competence requires procedural fluency and a certain level of conceptual understanding. Furthermore, they contended that strategic competence requires the ability to generate, evaluate and implement problem-solving strategies. That is, a student should generate possible problem-solving strategies. For example, to simplify the trigonometric expression $\frac{sin^2x+sinx-6}{sinx+3}$, a possible strategy is to factorise the numerator to $(sinx-2)(sinx+3)$ and then cancel the $(sinx+3)$'s to get $sinx-2$ as the final answer. However, strategic competence differs from procedural fluency in that strategic competence requires creativity and flexibility, since problem-solving strategies cannot be reduced to procedures. In this regard, Wilson and Heid (2011) referred to strategic competence as "knowing how" (p.8). Under the practical block of ATD, *knowing how* can be resolved by applying the correct technique to a task type.

2.2.3.2.4 Adaptive reasoning

Adaptive reasoning refers to students' ability to think logically about their work when dealing with concepts and operations (Kilpatrick et al., 2001). For example, to prove that $3tanx + \frac{3}{tanx} = \frac{3}{cosxsinx}$, a student with adaptive reasoning should be able to identify that it would be beneficial to prove the identity by starting on the right-hand side of the equation.

In their work with young children, Alexander, White and Daugherty (1997) proposed three conditions required for students to display reasoning ability. Firstly, the students should have enough *base knowledge*; secondly, the question must be *comprehensible and motivating*; and lastly, the setting must be *familiar and comfortable.* Similarly, Franke et al. (2009) agreed that for improved reasoning when dealing with word problems, students need meaningful mathematical dialogue. For example, when dealing with fractions, students sometimes erroneously add the numerators together and then add the denominators together: $\frac{1}{4}+\frac{1}{4}=\frac{2}{8}=\frac{1}{4}$. According to Alexander, White and Daugherty (1997), the students' base knowledge is lacking, thus adaptive reasoning cannot occur. This means, for example, that students with a lack of adaptive reasoning would find it difficult to simplify $\frac{1}{sinx}+\frac{1}{sinx}$.

Another indication of students using adaptive reasoning is their ability to justify their work. This study uses the same definition for the word ‘justify’ as Kilpatrick et al. (2001), i.e. to “provide sufficient reason for” (p.130). To ask a student to prove that two trigonometric identities are equal is equivalent to asking the student to ‘provide sufficient reasons’ why, for example, $tanxsinx + cosx = \frac{1}{cosx}$.

Adaptive reasoning intermingles with the other mathematical proficiencies, specifically in problem solving. Strategic competence provides a formula for solving a specific question, and adaptive reasoning provides the justification for that formula or procedures.

2.2.3.2.5 Productive disposition

Wilson and Heid (2011) contended that a student with a productive disposition believes that they benefit from a mathematical activity such as simplifying trigonometric expressions or proving trigonometric identities. Resnick (1988) warned that the term ‘disposition’ should not be misunderstood to infer any biological or inherent trait possessed by the student. Rather, ‘disposition’ in this sense refers to a ‘belief’ in being able to engage in higher-order mathematical thinking. This study sought to develop a sense of perseverance in students, through productive struggle. It used teacher intervention to support ‘getting started’, ‘carrying out a process’, ‘experiencing uncertainty in explaining and sense-making’ and ‘expressing misconception’; and thus pursued what Wilson and Heid (2011) called “persevering through multiple attempts to solve a problem” (p.8).

2.2.3.2.6 How does mathematics proficiency fit into ATD?

To solve any mathematical task (such as simplifying trigonometric expressions or proving trigonometric identities) requires mathematical proficiency. Thus, the type of technique a student might use to simplify a trigonometric expression or prove a trigonometric identity may rely solely on the student’s mathematical proficiency. For example, to prove that $tan^2xsin^2x = tan^2x - sin^2x$, a student who started to prove the identity from the left-hand side should have used adaptive reasoning after replacing tan^2x with $\frac{sin^2x}{cos^2x}$. This replacement would have yielded $\frac{sin^2x}{cos^2x}sin^2x$. However, the left-hand side of the identity contains a minus sign. This fact should have prompted the student to realise that either sin^2x should be replaced with 1-cos^2x, or that $cos^2x = 1 - sin^2x$.

The technology that underscores the technique students use to either simplify trigonometric expressions or prove trigonometric identities relies on conceptual understanding, procedural fluency, adaptive reasoning, strategic competence and productive disposition. The theory that justifies this technology is that mathematics proficiency is needed to successfully simplify trigonometric expressions or to prove trigonometric identities.

2.2.4 Teaching for Robust Understanding (TRU)

In this section this study used the *teaching for robust understanding* framework developed by Schoenfeld (2018) to justify why productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities, if correctly implemented, would yield students who are knowledgeable, disciplined, independent problem solvers and thinkers.

In his quest to unpack the very intricate topic of *teaching for robust understanding,* Schoenfeld (2017) posed the question: "What mathematics should we teach? (And is it possible to build a curriculum to do so?)" (p.2). Schoenfeld (2016) proposed a *teaching for robust understanding* framework that addresses impartial and robust learning environments and surroundings that make students more independent, flexible, knowledgeable mathematical thinkers. Schoenfeld argued that the quality of the learning atmospheres that teachers create for robust understanding depends on the following criteria: *mathematical content; cognitive demand; equitable access to content; agency; ownership and identity;* and *formative assessment.*

2.2.4.1 Mathematical content

When dealing with difficult mathematical content, students should have the opportunity to learn important mathematical content and practices, as well as having the disposition for problem solving. In the context of this study, in real-time teaching, it is notable that trigonometric expressions and identities are discussed only towards the end of the trigonometry chapter, which hints at the fact that students need to have accumulated some knowledge as a prerequisite for tackling the simplification of trigonometric expressions and the proving of trigonometric identities. The participating teacher in this study introduced only two trigonometric identities to the class, namely $sin^2x + cos^2x = 1$ and $tanx = \frac{sinx}{cosx}$. Furthermore, the teacher illustrated with only one example how to use these identities. Thus, when the students started on their activity questions relating to the simplification of trigonometric expressions, there was ample ‘doubt’ in the minds of the students (and sufficient opportunity for engagement) regarding what could or could not be used to simplify their trigonometric expression.

2.2.4.2 Cognitive demand

Schoenfeld (2018) contended that cognitive demand refers to the extent to which students have opportunities to grapple with and comprehend disciplinary ideas. He further stated that students learn best when presented with material ranging from moderate to challenging if the teacher supports them in their struggles. For this study, the ‘discipline’ that Schoenfeld (2018) refers to is mathematics. The disciplinary idea is the simplification of trigonometric expressions and the proving of trigonometric identities. According to Daily (2021), productive struggle is effective if the activity questions are difficult but not impossible. That is, the content must not be too easy, but neither must it be too difficult or beyond the students’ reach. Granberg (2016) stated that all students are likely to find themselves in a situation in which they experience productive and unproductive struggles. In the case of unproductive struggles, the teacher needs to offer support and or give feedback to these students. But the feedback does not necessarily have to come from the teacher. In a study conducted by Jonsson et al. (2014), the students did not get any feedback from the teacher during their problem-solving phase. However, these students received immediate feedback from a computer, and could thus adjust their problem-solving strategies. Daily (2021) contended that when teachers see students having trouble and help them, they reduce the cognitive demand and thus deprive them of productive struggles and the ability to make sense of difficult problems. To mitigate this situation for this study, regular meetings with the participating teacher were held to discuss the use of probing questions to maintain cognitive demand, and circumvent giving answers. When comparing probing questions asked by teachers with non-probing questions, Pierson (2008) found that probing questions that demanded higher-order thinking led to increased student learning and supported students’ productive struggles more.

Apart from highlighting the types of struggle the students are experiencing, probing questions – and questions in general, during the teaching and learning of trigonometry – can be used by teachers to respond to students' productive struggles. In addition, questioning can help students to draw connections between the definitions and rules used in the simplification of trigonometric expressions and the proving of trigonometric identities. For example, a student might be unsure if $sin^2x + cos^2x = 1$ implies that the square roots of both sides of the equation would yield $sinx + cosx = 1$. Here, the teacher might draw an analogy between the trigonometric identity and the application of the theorem of Pythagoras, $3^2 + 4^2 = 5^2$. In this regard, the teacher can show that if $sin^2x + cos^2x = 1$, this does not necessarily imply $sinx + cosx = 1$, just as $3 + 4 \neq 5$. Thus, in terms of ATD, the technology does not support the technique if $sin^2x + cos^2x = 1$, then taking square roots both sides of the equation yields $sinx + cosx = 1$.

2.2.4.3 Equitable access to content

Schoenfeld (2018) contended that equitable access to content refers to the extent to which classroom activity structures invite all students in the class to participate in these activities, with core disciplinary content such as simplifying trigonometric expressions and proving trigonometric identities being addressed by the class. The learning activities in this study were structured in such a way that from start to finish, the activities went progressively from simple to demanding tasks. That means the first activity question on the simplification of trigonometric expressions was relatively straightforward. However, as this study progressed, the activities became more difficult. The activities for proving trigonometric identities followed the same structure. Moreover, should the need have arisen, the teacher could have used scaffolding to assist the students.

All the students were first-language English speakers. Thus, any lack of communication because of a language barrier was eradicated. Additionally, all activities for the simplification of trigonometric expressions had a simple instruction: "Simplify the following expressions". Likewise for the proving of trigonometric identities, the instruction was: "Prove the following trigonometric identities".

2.2.4.4 Agency, ownership and identity

A student's *agency* refers to their capacity and willingness to participate in mathematical activities such as the simplification of trigonometric expressions and the proving of trigonometric identities. Schoenfeld (2018) referred to a student's agency as their chance to "walk the walk and talk the talk" (p.493). This 'walking the walk and talking the talk' gave students the opportunity to contribute to disciplinary ideas (such as simplifying trigonometric expressions and proving trigonometric identities) by building on other students' ideas and having other students build on theirs. By following this strategy, students were able to take ownership of the content and identify themselves as thinkers and learners (Schoenfeld, 2018).

2.2.4.5 Formative assessment

According to Schoenfeld (2018), 'formative assessment' refers to the degree that classroom activities elicit student thinking. For example, questions on the simplification of trigonometric expressions and proving of trigonometric identities should have been carefully designed to elicit common misunderstanding by students. Then the teacher could address these misconceptions by the students. In this regard the teacher could allow fellow students to explain to their peers how not to 'misunderstand' certain concepts, or the teacher could (what Schoenfeld (2018) calls) "meet the students where they are" (p.493) and formally re-explain certain concepts. For instance, in the simplification of the trigonometric expression $\frac{sin^2x-cos^2x}{sinx-cosx}$, students may (erroneously) divide sin^2x by $sinx$ and $-cos^2x$ by $-cosx$ and get $sinx + cosx$ as an answer. Here, the teacher may address the misconception by re-explaining the difference between two-squares factorisation and under what conditions cancellation can take place.

2.2.4.6 How does TRU fit into ATD?

To teach mathematics for *robust understanding*, the *task type* should support significant relationships between *procedures, concepts,* and *contexts*. For example, to simplify the trigonometric expression $tanxsinx - tan^2xcosx$ requires *procedural fluency* in factorisation. Next, a student should have the *conceptual understanding* to notice that $tanxsinx - tan^2xcosx$ and $tanx(sinx - tanxcosx)$ are not that different in terms of simplification. Thus, this task type would provide opportunities to engage in reasoning, problem solving and exchanging ideas. Consequently, the task type to teach for robust understanding contains elements of mathematical proficiency.

To use any *technique* to solve a given task that involves simplifying trigonometric expressions or proving trigonometric identities requires *cognitive demand.* When students engage in productive struggles to solve a task type, sufficient support from the teacher is required so that the technique employed to simplify the trigonometric expression or prove a trigonometric identity makes sense to the students. In this case *probing guidance* that is based on student thinking would encourage thinking and reasoning by the students.

Encouraging thinking and reasoning by the entire class ensures all the students will have *equitable access to mathematics.* This could lead to more creative and unique solutions by the students, which in turn could lead to students' *productive disposition.* Hence, equitable access to mathematics and productive disposition (as a technology) *justifies the technique* used by the students. The *theory* that justifies equitable access to mathematics by all students is that access to mathematics could lead to all students justifying their reasoning, and this could help the teacher identify any misconceptions regarding concepts.

2.2.5 Teacher noticing

This section discusses the role of 'teacher noticing' (or just simply 'noticing') during the teaching and learning of the simplification of trigonometric expressions and proving of trigonometric identities in high school. While the term 'noticing' in common everyday life situations refers to observations individuals make, in mathematics teaching it refers to how a teacher manages the "blooming, buzzing confusion of sensory data" (Sherin et al., 2011, p.5) that teachers are confronted with during instruction. In the teaching of the simplification of trigonometric expressions and the proving of trigonometric identities, the teacher is an active observer and facilitator capable of influencing the classroom discourses taking place. Supporting the adage that prevention is better than cure, Teuscher et al. (2017) analysed four pre-service teachers' journal entries (in the form of video observations) to help them prepare better for the classroom. Teuscher et al. (2017) expanded on Jacobs' (2010) definition of 'noticing' – that is, *attending* (listening*), interpreting,* and *responding* (talking) by dividing the terms into subsections:

‘Attending’ was divided into *general observation* and *student mathematical thinking.* Under ‘general observation’ no specific mathematical thinking was described, but broad descriptions of misunderstood mathematical concepts were observed. ‘Student mathematical thinking’ refers to observations that describe student mathematical inquiry.
‘Interpreting’ was divided into *general interpreting* and *root interpreting*. General interpreting is meant to describe broad generalisations of student confusion. Root interpreting refers to attempts to identify possible reasons for student thinking.
‘Responding’ was divided into three subcategories, namely *no clear connection, elaborated* and *facilitated*. Under *no clear connection* the teacher’s response is not connected to the student’s mathematical thinking. *Elaborated* refers to when a teacher responds to a student’s struggles by elaborating on the student’s existing knowledge, and *facilitated* means that the teacher facilitates further engagement in student discussion, to investigate the misunderstanding of mathematical concepts. This helps the teacher better understand any confusion and serve the student better.

However, the kind of guidance and structure teachers provide may either facilitate or undermine the productive efforts of students’ struggles (Tarr et al., 2008). On the one hand, Warshauer (2014) argues that explicit actions by teachers or classmates can work to build a community of understanding and supporting students’ productive struggles without depriving students of the opportunity to reason for themselves. On the other hand, in his investigation into cooperative learning for conceptual understanding of trigonometry, Rankweteke (2020) found that a lack of teacher instruction can lead to a deficit of deep understanding of trigonometry by the students.

But Daily (2021) contended that a teacher’s desire to help struggling students can result in lowering or removing the cognitive demand required by the mathematical task. In this study, the researcher and the teacher met on a regular basis to either discuss what had transpired during some of the activities, or plan for possible questions or struggles the students might encounter. Of the three noticing skills, Jacobs and Ambrose (2008) reported that learning to respond to children’s thinking and/or productive struggles is the most difficult skill for teachers to develop. However, Jacobs and Ambrose (2008) noted that responding to student thinking or productive struggles is the most effective way to extend student learning. Nevertheless, Chao et al. (2016) contended that teachers can develop questioning and listening skills over time, but the ability to respond does not automatically develop with experience. In this regard, Jacobs, Lamb and Philipp (2010) defined *professional noticing* as teacher noticing that not only includes teachers' attention to and interpretation of classroom activities, but also how the teacher plans to respond to those activities.

But in this study, the teacher noticing was clear when students struggled with simplifying trigonometric expressions and proving trigonometric identities. Chao et al. (2016) warned the researcher to keep an open mind, since teacher noticing can be skewed: teachers sometimes judge students, whether consciously or unconsciously, according to their gender, ethnicity, mathematical knowledge and even dress code.

2.2.5.1 Types of interactions between teacher and students

This section discusses the interaction between teacher and student. This interaction or lack thereof determines the effectiveness of teacher support when students struggle with the simplification of trigonometric expressions and the proving of trigonometric identities.

Rowe (1974) coined the term 'wait time', which refers to the time between when a teacher poses a question and when a student responds to the question, and the time between when the student has responded until the teacher gives the student feedback. Rowe (1974) also discovered that teachers typically wait only one and a half seconds for a student to respond. She found that to ensure better attitudes and behaviour from both teachers and students, wait time should be extended to at least three seconds. Rowe (1986), Stahl (1990) and Tobin (1987) reported benefits resulting from increasing wait time to over three seconds: the correctness and length of responses increased, the number of 'I don't know' and no-answer responses decreased, student participation with appropriate answers increased, student academic achievement increased, student questioning strategies were more diverse, there was an increase in the number of quality questions and a decrease in the number of queries, and teachers asked more in-depth questions that required higher-order thinking.

These are the types of interactions that occur between a student and a teacher: *Initiation response feedback* happens when the teacher poses a question (initiation) followed by an answer (response) from the student, who then receives feedback from the teacher. For example, the teacher might ask the class what the best strategy is to prove the trigonometric identity $tanxsinx + cosx = \frac{1}{cosx}$. In return, the class might respond with: "Let's simplify the left-hand side." The feedback from the teacher that follows might be: "Why would it not be beneficial to start on the right-hand side?"

This type of response was identified, during noticing, more than three decades ago. Stigler and Hiebert (1999) contended that its pervasiveness in the classroom cannot be underestimated. However, as in the example above, if a correct justification for the teacher's feedback was not given (i.e. why it would not be beneficial to start on the right-hand side), this type of interaction between teacher and student could lead the student into a predetermined way of thinking (Cazden, 1988; Nystrand, 1997).

In the 'funnelling' process, students are guided through a procedure to solve a problem through a series of questions, as explained by Wood (1998). Although most trigonometry identities are proven starting from the left-hand side, this process is not always feasible; thus, funnelling would be an appropriate interaction.

Unlike funnelling, in which guidance to solve a problem is predicated on teacher thinking, Wood (1988) explained that *focusing* is when guidance to solve a problem is grounded in student thinking. This method requires deep-seated subject and content knowledge from the teacher, since he or she needs to follow and understand the student's way of thinking. It also shows that the teacher appreciates the student's thinking, thus inspiring the student to contribute to classroom activities. Cazden (1988) argued that for students to become independent learners, the questioning that accompanies focusing should eventually be reduced. For example, when asked to prove that $\frac{1}{sinx-1} + \frac{1}{sinx+1} = \frac{-2sinx}{cos^2x}$, a student might respond by simplifying the left-hand side of the equation and get $\frac{2sinx}{sin^2x-1}$. The student might exclaim: "I don't know how to proceed!" A focusing question from the teacher might be: "Well done! I see that your numerator mimics the right-hand side numerator – how can we make the denominators the same?"

Herbel-Eisenmann and Breyfogle (2005) argue that to turn funnelling into focusing, the teacher should repeat important information about a problem and rephrase prominent features of the students' work. During this rephrasing of student solutions, the teacher should show that he or she values the language that the student used in their solution – by 'language', this study refers to both written and oral language, and the mathematical language of symbols. This strategy helps students to articulate their own arguments and understand their fellow classmates' reasoning.

Once teachers notice a student struggling, they should decide how to react (Jacobs et al., 2011). For example, to prove the trigonometric identity $cosxsin^2x - cosx = -cos^3x$, a student might incorrectly factorise by writing $cosx(sin^2x - cosx)$ and struggle to continue with the solution. Although the student's action was incorrect, the teacher could have used this moment to use Herbel-Eisenmann and Breyfogle's (2005) suggestion to rephrase the student's solution and guide him or her into correct factoring. Of importance here is for the teacher to acknowledge to the student that their choice of method was correct, but that its execution was not accurate. The teacher's action would have coincided with the opinion of Maybin et al. (1992), who asserted that one category of teacher response is to provide information to the struggling student.

Although this type of action makes students more task-enabled than before the interaction, such a response may affect a task's level of cognitive demand (Maybin et al.,1992). The teacher could have said to the students: "Right – but if you just throw parentheses in there randomly, you are changing the problem; what can we do with parentheses that will lead us?" This question increases the cognitive demand on the students since they must try to understand the teacher's point of view.

Teacher responses with examples that connect students' thinking to their prior knowledge can give students useful strategies and skills for approaching the task at hand. Using this technique, a teacher could ask students what their first step would be in factorising a problem such as $3x^2 + 2x$. This would agree with Richland et al. (2004), who claimed that using analogies is another strategy teachers may use to effectively connect students' productive struggles with elements of their prior knowledge. O'Connor and Michaels (1993) echoed this sentiment, saying that teachers' use of re-voicing is a useful tool to help students connect to prior knowledge. So, teacher noticing that leads to carefully sequenced questions to develop and build on students' ideas can give an indication of the students' mathematical reasoning, and thus help students focus on mathematical points that they might otherwise have missed (Anghileri 2006; Cazden 2001).

In noticing, teachers should be wary of the language they use; statements such as 'that is totally wrong' can cause students to feel that their efforts have been discarded (O'Connor & Michaels, 1993; Van Zee & Minstrell, 1997). As an alternative, Doerr (2006) asserted, statements such as 'you're on the right track' can give students a new sense of confidence and agency.

These actions would lead to the creation of what has been referred to in TRU as *agency;* as well as *competence,* which is a crucial part of how students use opportunities to learn and contribute to conversations about disciplinary discourses (Gresalfi et al., 2009; Schoenfeld, 2018). Duckworth (2001) and Empson and Jacobs (2008) argued that listening is at the centre of effective mathematics teaching practice. Consequently, Stein and Smith (2000) and Stein et al. (2008) reasoned that by anticipating students' responses, teachers can learn how students understand and approach a mathematical question with appropriate or inappropriate strategies, and possibly avoid any misconceptions that may arise.

2.2.5.2 How does teacher noticing fit into ATD?

Teacher noticing lies firmly in the didactic praxeology. If a student struggles with a question and uses an incorrect strategy in attempting to solve it, through noticing and interpretation of the student's strategy the teacher can advise the student on how to approach the question differently. Hence the 'attending' and 'interpreting' description of noticing would make up the *task type* under ATD for teacher noticing. For example, when proving trigonometric identities, a student must try to prove that the left-hand side of an equation is equal to its right-hand side. Thus, the student's starting point may well be the left -hand side of the equation. However, if the teacher notices the student's strategy, he or she could suggest that starting on the right-hand side of the equation might be an easier option. The type of guidance that the teacher gives a student, and hence how the teacher *responds* to a student, can be considered the *technique* under ATD for teacher noticing. For example, to prove the trigonometric identity $cos^2xtan^2x + cos^2x = 1$, a student might erroneously ignore the tan^2x and assume that $cos^2x + sin^2x = 1$. The (noticing) technique that the teacher could employ is to ask the student: "What happened to tan^2x?".
The *technology* under ATD that justifies the *technique* employed by the teacher under noticing is to address any misconceptions that the student might exhibit.

Lastly, the *theory* under ATD is that effective teacher noticing can assist students when they go through productive struggles.

2.2.6 Teacher questioning

In this section, teacher questioning is introduced with reference to the simplification of trigonometric expressions and the proving of trigonometric identities. While questioning in the classroom is ubiquitous, in the context of this study the main purpose of teacher questioning was to stimulate student thinking to evoke prior problem-solving strategies established by the students to simplify trigonometric expressions and prove trigonometric identities, as proposed by Wood and Anderson (2001) and Mason (2020). However, Martino and Maher (1994) reported that: "The art of questioning may take years to develop, for it requires an in-depth knowledge of both mathematics *[in the case of this study, the simplification of trigonometric expressions and proving of trigonometric identities]* and children's learning of mathematics. Once acquired, the teacher has available a powerful tool to support students in their building of mathematical ideas". (p.2). Thus, this study agrees with Brualdi (1998) that questioning must be very specific, to elicit precise information that will help students simplify trigonometric expressions and prove trigonometric identities, and thus create a conducive atmosphere for problem solving.

As Brualdi (1998) asserted, to teach well, teachers must ask probing questions, so that a successful interaction can occur between the teachers and their students. Furthermore, Morgan and Saxton (1991) listed the following benefits of questioning: firstly, questioning by the teacher helps to keep students focused by actively involving them in the activity, for example in simplifying trigonometric expressions and proving trigonometric identities. Secondly, students can reciprocate, expressing their ideas and thoughts by answering the questions. For example, to prove that $\frac{1-cosx}{sinx} = \frac{1}{sinx} - \frac{1}{tanx}$, the teacher might ask the students: "Since $1 - cos^2x = sin^2x$, does it mean that $1 - cosx = sinx$?" The students might reciprocate by replying: "Why not, teacher? It works for $x = 0$", thus starting an exchange of ideas between the teacher and the students. Thirdly, by expressing their ideas, fellow classmates can acquire different perspectives on the activity. For example, regarding whether $1 - cosx = sinx$ in the preceding sentence, a fellow classmate might state: "Wait! The equation does not work for $x = 45^0$." Fourthly, questioning can assist teachers to pace their lessons and manage class behaviour. In this regard, questioning can serve as a strategy to keep students focused on the simplification of trigonometric expressions and the proving of trigonometric identities. Lastly, teachers can evaluate student understanding of a concept (such as the simplification of trigonometric expressions or the proving of trigonometric identities) and revise and evaluate their lesson appropriately.

Questioning not only evaluates students' mathematical knowledge, but shows how students think Aizikovitsh-Udi & Star, 2011); also, good questioning techniques allow students to articulate their thoughts better (Stein & Smith, 2000). In their investigation into what constitutes 'good questioning' in the mathematics classroom, Aizikovitsh-Udi and Star (2011) identified two types of teachers: the *conserving teacher* versus the *leveraging teacher*. Conserving teachers see themselves as being central in the classroom. They give sufficient explanations with direct instruction, serve as the source of mathematical knowledge, and thus decide the correctness of answers; they ask questions, but answer most of the questions themselves; and lastly, conserving teachers do not allow discussion.

By contrast, the leveraging teacher's role is also central, but with a different emphasis. Leveraging teachers guide students with few rules and little assistance, allowing them to choose different paths to solve a problem. Leveraging teachers do not answer their own questions, but allow the students ample time to respond; they encourage discussion, and are actively engaged in the classroom by altering their questions for students who do not reply. By altering their questioning, leveraging teachers evaluate students' mathematical content knowledge and how they think. In this study, the researcher and the participating teacher met on a regular basis to plan for any possible scenarios which included possible questions from students. They came to the understanding that to implement productive struggles successfully, an absolute minimum of the conserving teacher approach should be used; the teacher should lean more towards the leveraging approach.

Teachers should take care when questioning students; Shahrill (2013) agrees with Carter (2008) and Kapur (2011) that students should not feel pressured to answer questions, but do so freely, and that 'wrong' answers should not be seen as failures, but as prospective discoveries, in order to better support and encourage students to persist in problem solving. As a next step, this study sought to explore the types of questions posed by the teacher in the classroom. By investigating these question types, this study could then analyse which of these types promote or stifle productive struggles. Boaler and Brodie (2004) proposed nine types of teacher questions. In Table 2.4 below, these nine types are illustrated with their associated characteristics. An example of each question type is also supplied. The facilitating teacher was made aware of some of these question types at the beginning of the research process.

Table 2.4: Types of questioning by the teacher

Question Type	Characteristics	Example
1. Collecting information	Requires students to state facts or known procedures. Immediate response from student is required.	What is the definition of *sinx?*
2. Incorporating terminology	Enables students to us terminology correctly when a discussion is underway.	Just like $3x^3 + 5x^2 - 9$ is called an algebraic expression, $5sin^2x + 2sinx + 4$ is called a…expression?
3. Exploring mathematics meanings and relationships	Indicates mathematical meanings and relationships. Relates mathematical ideas and representations.	We can factor out $x$as a common factor in $x^2 - xy$. Similarly, what can we factor out in $sin^2x - sinxcosx$.?
4. Probing	Requires students to explain and articulate their ideas.	To prove the trigonometric identity $\frac{6sin^2x+5sinx-6}{3sinx-2} = 2sinx + 3$, a student might proclaim that the answer is $2sinx + \frac{5}{3} + 3$. The teacher could ask the student to explain his or her answer.
5. Generating discussion	Asks for contributions from different members of the class.	Student: Would you put this over 1? *[meaning* $\frac{cosx}{tanx}.\frac{sinx}{1}$*]* Teacher: What do your classmates say?

6. Linking and applying	Indicates the relationship between mathematical ideas and other areas of study.	Questioning by the teacher can relate the factorisation of algebraic expressions to the factorisation of trigonometric expressions.
7. Broadening thinking	Broadens the discussion topic to other relevant situations.	If $\frac{2}{x}+\frac{3}{y}=\frac{2y+3x}{xy}$, then what would $\frac{2}{sinx}+\frac{3}{cox}$ be?
8. Orienting and focusing	Assists students to focus on essential elements of a situation, to help them with problem solving.	When we simplify $\frac{x^2+2x+1}{x+y}$ what would be best to do with the numerator?
9. Forming context	Discusses topics outside of mathematics to show how links can be made with mathematics.	What type of trigonometric graph does the vibration of a violin string resemble?

Probing questions from the teacher correspond with probing guidance from teacher responses to productive or unproductive struggles. That is, the questioning and the guidance from the teacher are based on student thinking. In a similar way, affordance under teacher responses to productive or unproductive struggles also encourages student thinking; hence, questioning is student-centred.

2.2.6.1 Questioning behaviour that increases student learning

Wilen (1987) proposed the following eight questioning behaviours to increase student learning. The teacher should state *clear questions*; to eradicate any student miscues or confusion, the teacher must employ unambiguous questions. For example, instead of asking the students "Where does $1-cos^2x$ come from?", he can rephrase the question by asking: "If $sin^2x+cos^2x=1$, how can we manipulate that equation to get $1-cos^2x$?" Questions should be of an *academic* nature; non-academic questions – that is, questions that do not pertain to the subject matter – may influence the social-emotional climate in the class, but have been shown not to increase student achievement. The teacher should pose *frequent factual questions*; for example, the teacher might ask the class: "What is sin^2x+cos^2x

equivalent to?" *High-cognitive questions* should be directed to cognitively mature students who can reason abstractly.

Students should be afforded *enough time to respond*. Ingram and Elliott (2016) stated several advantages to teachers waiting more than three seconds for an answer after asking a student a question; *students give longer responses*, which *may contain logical reasoning and explanations. Students can justify more of the conclusions they reach*. T*he incidence of theoretical thinking increases. Student questioning increases*. *More interaction occurs*, and there is a *reduction in the teacher being seen as the centre of learning*. The *students fail to respond less often*. There is *an increase in student discipline* in the classroom. And lastly, *voluntary participation and student confidence increases, and student achievement improves.* Dillon (1988) recognised that the teacher should pose questions that invite a wide spectrum of students to participate, thus increasing student-to-student interaction and allowing the students to take ownership of the lesson.

2.2.6.2 How does teacher questioning fit into ATD?

Teacher questioning lies firmly under didactic praxeologies. In particular, the teacher questions students to elicit prior knowledge that the student may have that will assist them to create a technique for solving a given task. Thus, when students struggle to solve a task that involves the simplification of trigonometric expressions or the proving of trigonometric identities, through questioning the teacher can gauge where the student's deficiencies lie. For example, if a student struggles to solve the sum of two rational trigonometric expressions, the teacher might notice through questioning that the technique that the student is employing is correct, but that the student lacks knowledge about adding fractions.

From an ATD perspective, the student's deficiency in the technology that justifies the technique – in this case, the adding of fractions – prohibits the student from completing the task. Furthermore, since the teacher has the necessary subject knowledge, he or she can recognise the theory that justifies the technology; in this example, that fractions are the gateway to higher mathematics. This means that if students have a weak background in fractions, they are likely to experience challenges in understanding concepts in algebra, higher-order mathematics (such as simplifying trigonometric expressions and proving trigonometric identities) and calculus (Aliberti, 1981). Mhakure et al. (2014) echo the sentiment that although proficiency in fractions does not guarantee success in algebra, deficiency in the concepts of fractions will hinder progress in simplifying algebraic fractions; and by extension, the simplification of trigonometric expressions and proving trigonometric identities.

2.2.7 Newman Error Analysis (NEA)

The purpose of applying the NEA to the students' assessments in the activities is to establish whether or not there is a link between the errors the students make when they simplified trigonometric expressions and proved trigonometric identities in the assessments, and the errors the students make during their productive struggles in real time during activities.
Alhassora et al. (2017) described Newman's five error types as:

1. read and record,
2. comprehension,
3. transformation,
4. process skill, and
5. encoding.

Read and record refers to a student's ability to read a mathematical question and determine the meaning of the words and symbols in the question. Comprehension remarks on a student's understanding of the symbols, expressions and words in a mathematical question. Transformation describes a student's ability to choose the correct formula to solve a mathematical question. The process skills refer to a student's skill in evaluating whether a process to solve a mathematical question was correct or not. Lastly, the encoding error type evaluates whether a student can justify a solution to a mathematical question.

Septiandi et al. (2020) used NEA to note the errors that students make in trigonometry. They then used these errors to analyse why students perform poorly in calculus. By breaking down errors and misconceptions, teachers can manipulate tasks so that students can learn from their mistakes (Anthony & Walshaw, 2009). For example, to prove that $\frac{sin^2x-4}{sinx-2} = sinx + 2$, a student might (erroneously) argue that the left-hand side can easily be manipulated by dividing sin^2x by $sinx$ and -4 by -2. Although the student's erroneous thinking will yield the same (correct) result, changing the trigonometric identity to prove that $\frac{sin^2x-5sinx+6}{sinx-2} =$

$sinx - 3$ may put doubt in the student's mind on how to deal with the middle term in the numerator.

Abdullah (2015) suggested that students follow a five-stage (five error types) hierarchy when solving one-step word problems. Firstly, they should *read the problem* – this entails reading the mathematical problem and understanding the sentence and the mathematical symbols used. In this study, the instructions to solve the mathematical problem were clear: 'Simplify the following trigonometric expression' or 'Prove the following trigonometric identity'. The next stage in the hierarchy is to *comprehend* what was read. This means that the students should understand what they have read. Following this is the *transformation* stage, in which the student selects an appropriate mathematical strategy to solve the problem. For example, to simplify a trigonometric expression or to prove a trigonometric identity, the student might choose factorisation, or the distributive law, as a strategy to solve the problem. Next is the *process skill* stage. In this stage the student should perform the necessary process to simplify the trigonometric expression or to prove the trigonometric identity. For example, to simplify $tan^2x\left(cos^2x + \frac{cos^4x}{sin^2x}\right)$, if a student performs the distributive law first then he or she should write $tan^2xcos^2x + tan^2x\frac{cos^4x}{sin^2x}$. The *process skill* here is the execution of the distributive law. Lastly, the *encoding* stage requires students to represent their answers in a mathematical way.

Students do sometimes solve mathematical questions through dubious methods; but Newman (as cited by Abdullah, 2015) suggested that the word 'hierarchy' is there to inform the reader that failure at any level will result in getting the incorrect answer. For example, if a student tries to simplify the trigonometric expression $\frac{sin^2x+cos^2x}{sinx+cosx}$ and erroneously gets $sinx + cosx$ as an answer, then he or she has incorrectly performed the process of manipulating algebraic fractions.

Numerous researchers have used NEA on problems other than word problems. Zakaria (2010) used NEA to study errors committed by Indonesian secondary school students when solving quadratic equations, reporting that students mostly make mistakes at the transformation and process skill stage of the hierarchy. Mensah (2017) used the NEA model to analyse Ghanaian senior high-school student errors in learning trigonometry. Although

Mensah (2017) listed ‘carelessness’ as a type of error, this study agrees with Clements and Ellerton (1996), who argued that carelessness and lack of motivation can occur as part of any of the five error types. Mensah (2017) reported that when students were required to solve trigonometric ratios using formulae, most errors occurred during the process stage, followed by the transformation stage. Riastuti et al. (2017) also used NEA to decode errors made by Indonesian geometry students. The researchers divided the students into groups of low, medium and high spatial intelligence. They concluded that comprehension errors occurred mostly in those with low spatial intelligence ability, followed by students with medium and high spatial intelligence; errors made regarding transformation, process skill and encoding mirrored these results. And based on cognitive styles, Zamzam and Patricia (2018) used NEA to decipher the errors made by four students of mathematics education teaching.

Cognitive-impulsive children tend to answer swiftly, but are not meticulous, and often give an incorrect answer; as opposed to cognitive-reflexive children, who are slow to give answers but whose answers are more thought-provoking and usually correct (Warli, 2013). Irrespective of cognitive mindset, Warli (2013) found that these students made the most errors at the transformation and process skill hierarchy.

Saleh et al. (2017) used NEA to analyse the mistakes made by 148 Indonesian high-school students when using analogical reasoning. This means the students were required to make associations between two questions: one a given question, also known as the source, and a second question referred to as the target. The source question in this case was: ‘Find a solution to the equation $x^2 + 5x + 6 = 0'$. The target question was: ‘Find a solution to the equation $\cos 2x + 6sinx + 7 = 0'$. The researchers concluded that *reading errors* occurred because students could not see that $cos2x = 1 - 2sin^2x$; *comprehension errors* occurred because the students could not transcribe from the source question to the target question; *transformation errors* occurred because students could not transfer trigonometric equations to general quadratic equations – for instance, if $sinx$ is replaced with a, then with the appropriate trigonometric substitution of $cos2x$, the equation $cos^2 2x + 6sinx + 7 = 0$ would be analogous to the quadratic $-2a^2 + 6a + 8 = 0$. *Process skills* errors occurred because students could not solve the target question by means of a quadratic equation and making applicable changes; for example, students might incorrectly factorise $-2a^2 + 6a + 8$ as $(2a + 4)(-2a + 2)$. *Encoding errors* occurred because students could not find an answer in degrees or radians for the target question. As alluded to earlier, careless errors were made throughout the different Newman phases (Saleh et al., 2017). Similarly, Sartika and Fatmanissa (2020) used NEA to identify Indonesian students' errors in trigonometry that assessed higher-order thinking skills.

To identify the type of error the student is making, the teacher must follow the next procedure in the NEA: to identify a reading error, ask the student to read the question to you, and to let you know if there are any words in the question they do not understand. For a comprehension error, ask the student what the question wants them to do. To identify a transformation error, ask the student which method they employed to find the answer. For a process skills error, ask the student to verbalise their work, explaining each step. Lastly, to identify an encoding error (an inability to express the answer in acceptable form), ask the student to tell you his or her answer, and to point to it. If the student gets the correct answer at this second attempt and you are convinced after listening to the student's responses to the NEA that he or she made a mistake in the first attempt to solve the question, the error is classified as a careless mistake.

Similarly, Polya (1985) stated that problem solving involves four steps: firstly, you need to understand the problem. Secondly, set up a plan for completion. Thirdly, solve the problem according to the plan. Lastly, re-examine, to see that all steps have been done. This corresponds closely with Newman's error-analysis process. Prakitipong and Nakamura (2006) asserted that if a student does not make any reading or comprehension errors, this signifies that the student interpreted the question correctly. This correlates to Polya's understanding and planning for completion of a problem. Prakitipong and Nakamura (2006) also contended that if a student is successful in terms of the last three NEA error types – that is, *transformation*, *process skills* and *encoding* – this indicates that the student was successful in solving the problem. This correlates with Polya's instruction to attempt the problem according to the plan, and check that all steps have been done.

2.2.7.1 How does Newman error analysis fit into ATD?

All the researchers discussed above have recommended that teachers use the errors committed by students to adjust their teaching style in order to address the errors. It is precisely this strategy that the participating teacher in this study used to adapt his questioning techniques to promote strategic learning when students in the study struggled with simplifying trigonometric expressions and proving trigonometric identities. In terms of ATD, if a student employs an incorrect technique in simplifying trigonometric expressions and proving trigonometric identities, this indicates to the teacher the type of error that the student is making. Based on this information the teacher in this study immediately altered his questioning, guiding his student in the right direction. The alternate guiding or questioning is based on the technology that justifies the technique. Similarly, when the researcher observed errors during the focus group interviews, he could adjust his questioning; however, in this instance his questioning was targeted to investigate the misconception that caused the errors. Lastly, errors were also observed during the scrutiny of students' scripts. However, these errors were used to inform the teacher about how to address any misconceptions regarding mathematical concepts.

2.3 Summary

Chapter Two presented a general view of ATD – how ATD positions itself in mathematics education, and its relevance to the current study.

The constructs *productive struggles, mathematical proficiency, teaching for robust understanding, teacher noticing and questioning,* and *NEA* are all important components of an effective teaching strategy. Furthermore, this study has illustrated that these constructs are interdependent. For example, to struggle productively, students need some mathematical proficiency. On the other hand, to teach for robust understanding, students should be afforded the opportunity to grapple with questions that are within their reach to solve with the help of the teacher. However, through teacher noticing and teacher questioning, the teacher can identify any misconceptions the students might have that could stifle their progress in their struggles. Additionally, the use of NEA could expose productive struggles that remained from the learning activities to the assessments. Subsequently, teachers can use this result of the remaining productive struggles to achieve better teacher questioning and noticing to address these productive struggles.

In Chapter Three, the research design and methodology are explored.

CHAPTER THREE: RESEARCH DESIGN AND METHODOLOGY

3.1 Introduction

Chapter Three lays out the research design and methodology for this study. It begins by discussing the research design and research paradigm. The research paradigm gives rise to a discussion of qualitative research methodologies and presents a case for descriptive statistics supplementing this methodology. Next, this chapter unpacks the research instruments and explains the activities, assessments and interview guides used in this study. The research procedures are then described by sketching the research site, how the sample of participants was selected, the data collection process used, the role of the participating teacher, the role of the researcher, how the reliability and validity of data were secured, and the minimisation of researcher bias. Details of the qualitative data analysis are presented, which is followed by how the important issue of ethics was addressed, since minors were involved in this study.

Finally, there is a summary of Chapter Three; and an explanation of how all the aspects considered in this chapter lead to Chapter Four, and its analysis of the themes from the video recordings, assessments, interviews and error analysis.

This study used qualitative research methodology with descriptive statistics. The qualitative research methodology was used to gather data via video recordings of the activities, and audio recordings of the focus group interviews and teacher meetings. To analyse the assessments that the students completed after the activities, descriptive statistics was used.

3.1.1 Research design (conceptualisation of the study)

The chapters of this study illustrating the research process are not mutually exclusive. Although each chapter can be read as a standalone topic, the reader is advised to stand back and see each element as being a building block that contributes towards an overall process. Firstly, the researcher decided to investigate the nature of the productive struggles experienced by students during the simplification of trigonometric expressions and the proving of trigonometric identities. The researcher divided this study into two corresponding sections, namely the simplification of trigonometric expressions and then the proving of trigonometric identities. For ease of comparison, both sections use only the ratios $sinx, cosx\ and\ tanx$. Secondly, an extensive literature review was conducted to identify the most appropriate theoretical frameworks. Thirdly, based on the selected topic and the available literature, various research questions were proposed, as stated in Chapter One. Fourthly, the research design and research methodology were identified and refined, as proposed in Chapter Three and illustrated in Figure 3.1 below.

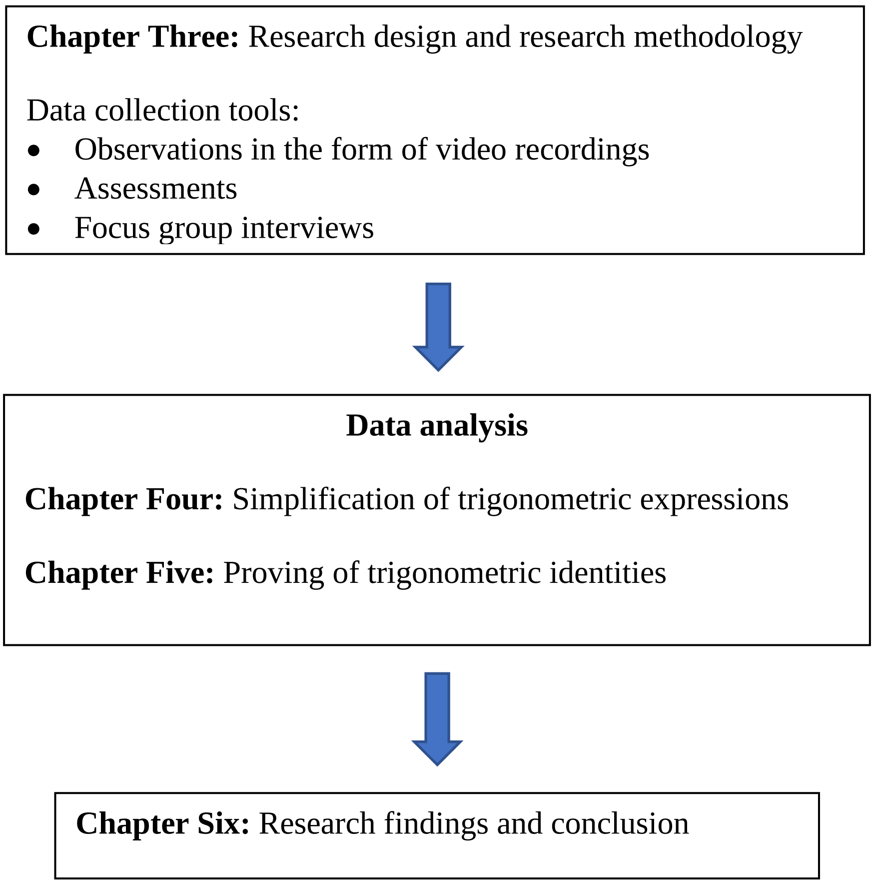

Figure 3.1: Research design for this study

This is followed by the creation and use of the relevant data collection tools in Chapter Three, and the analysis of the data obtained in Chapters Four and Five. Finally, the research findings and conclusions are discussed in Chapter Six.

3.2 Research paradigm

In this section, this study strove to explain its research paradigm. Perera (2018) described a research paradigm in educational research as the researcher's viewpoint or ideology. This view, Kivunja and Kuyini (2017) argue, informs the meaning or interpretation of the researcher's data. Kumatongo and Muzata (2021) contended that a research paradigm is based on philosophies; namely epistemology, ontology and axiology. This study uses interpretivism as its research paradigm.

Epistemology of a Paradigm

The epistemology of a paradigm concerns itself with how knowledge is obtained or communicated to others (Cooksey & McDonald, 2011). Interpretivism epistemology is subjective. For this study, the researcher is a mathematics teacher that has taught the subject for 29 years. Rehman and Alharthi (2016) contend that interpretivism research does not discover context free knowledge and truth but tries to interpret individual action within a social context. This study took place as part of the normal trigonometry curriculum. and the knowledge obtained is the simplification of trigonometric expressions and the proving of trigonometric identities. *How it is communicated* is by means of productive struggles.

Ontology of a paradigm

Scotland (2012) has stated that ontology is a branch of philosophy that deals with the assumptions we make to make sense of our reality. Also, Kivunja and Kuyini (2017) contended that ontology deals with the essence of the social phenomenon that a study is investigating. In this study, the essence of investigating productive struggles when students simplify trigonometric expressions and prove trigonometric identities is to improve students' general understanding of these concepts, as well as to improve teaching strategies when teachers explain the simplification of trigonometric expressions and how to prove trigonometric identities. In addition, Rehman and Alharthi (2016) reasons that knowledge is a consequence of interpretivism ontology.

Axiology of a paradigm

Axiology refers to the ethical issues that must be considered for the study. Since minors were involved in this study, ethical clearance was sought and obtained from the University of Cape Town and written consent was sought and obtained from AWS High School in Chesterfield County, South Carolina, USA, and from the participating students' parents.

Methodology of a paradigm

Kivunja and Kuyini (2017) argued that the methodology of a paradigm includes but is not limited to data gathering, sample participation, instruments used, and data analysis. Rehman and Alharthi (2006) argue that interpretivism methodology sees any social phenomena in context. In the case of this study, all interactions took place as part of the regular trigonometry curriculum. The data from the focus group was analysed through an inductive method. This means that themes and patterns were sought from the audio recordings of the focus group.

The rest of Chapter Three explains in detail the methodology used for this study. This study uses qualitative research design with descriptive statistics. What follows is an explanation of the advantages of this choice.

3.2.1 Qualitative research approach

Qualitative research is the analysis of written, visual and audio data (Mihas, 2019), which aligns with the contention of DeFranzo (2011) that qualitative research is investigative in nature, and that it can be used to gain an understanding of a phenomenon, as well as defending the investigation. Also, qualitative research is ideally suited to capturing social aspects such as the support that a teacher provides to students during their productive struggles (Greene, 2007).

Qualitative research design also seeks to understand people's emotions, experiences, attitudes, convictions and interactions, by generating non-numeric data (Pathak, et al. 2013). It refers to people's lived experiences; in the case of this study, the productive struggles experienced by students when simplifying trigonometric expressions and proving trigonometric identities. It can thus produce detailed descriptions of students' feelings, emotions and frustrations when they struggle with mathematical tasks. Furthermore, student behaviour in the classroom may be influenced by factors outside of the research focus, which qualitative research is able to identify (Rahman, 2017). Moreover, qualitative research can give a comprehensive account of the teacher support given to the students during their productive struggles, and can also analyse the culturally specific support the teacher gives to the students; in other words, support that is acceptable in American society, but which may be perceived as unacceptable in other cultures. For example, in American society, deliberately allowing students to struggle can be seen as neglecting your job as a teacher, whereas other cultures such as Japan embrace this type of struggle. Thus in America, teachers are more likely to assist students in their struggles, allowing students fewer opportunities to figure out the learning of mathematical concepts.

Between 1995 and 2005, qualitative methodology was the dominant mode of research in mathematics education, accounting for fifty percent of the 710 articles surveyed by Hart, Smith, Swars and Smith (2009) when they compared the use of qualitative, quantitative and mixed research methodologies. However, Hart et al. (2009) also reported that mixed methods were the second most popular research methodology in mathematics education during the same period. Ross and Onwuegbuzie (2012) had similar findings, confirming that qualitative research was more prevalent than quantitative research in mathematics education. This study continues this trend, using a qualitative research methodology supplemented with descriptive statistics. Also, this study took place in a classroom used by the students and the participating teacher for their regular instruction, conforming with Noble and Smith's (2014) recommendation that qualitative studies should take place in a natural environment.
The qualitative research methodology was used predominantly to gather data, via video recordings of the activities as well as focus group interviews. At the end of each DCAT level an assessment was issued to the students. Also, by using NEA the researcher could ascertain which productive struggles had not been resolved when the students took the assessments.

Qualitative research has limitations, due to its small sample sizes; this was also the case in this study. This raises issues of generalisability – in other words, for this study, whether the findings are generally applicable to all students who experience productive struggles in simplifying trigonometric expressions and proving trigonometric identities (Harry & Lipsky, 2014; Thomson, 2011). Nonetheless, in the educational setting, qualitative research is becoming more prominent (Rahman, 2017). However, for this study, descriptive statistics were used as a supplement to the qualitative research approach.

3.2.2 Descriptive Statistics

Kaur, Stoltzfus and Yellapu (2018) defined descriptive statistics as a method for summarising data in an organised way to describe the relationship between variables in a sample. Descriptive statistics include the following types of variables: nominal, ordinal, interval and ratio. Additionally, descriptive statistics include central tendency, and measures of frequency, position, and dispersion/variation. This study used descriptive statistics to illustrate, in table form, the relationship between the distribution and types of productive struggles, and the activity questions for the simplification of trigonometric expressions and the proving of trigonometric identities. Furthermore, a similar platform was used to describe the relationship between the tally and type of NEA, and the activity questions. This study used statistics to describe the percentage increase or decrease in errors committed by the students between assessments when simplifying trigonometric expressions and proving trigonometric identities. Also, statistic was used to calculate the percentage of NEA errors as a ratio of the total errors committed at each DCAT level.

3.3 Research instruments

The research instruments – that is, the ‘tools’ used to gather data for this study – are explained in this section.

3.3.1 Activity questions on simplification of trigonometric expressions

This study took place in the context of a broader curriculum for the students at the participating school. That is, no special provisions were made by the school for this study. Consequently, this study commenced when the teacher started with the simplification of trigonometric expressions as part of his teaching pacing guide for the year. Apart from the DCAT levels, the selections and source of the activity questions were informed by the curriculum. The source of the questions was the prescribed textbook. This natural approach, optimising the relevance of research to practice, also helps to improve the external validity of the research findings, since it is impossible to fully account for external validity (Findley, Kikuta & Denly, 2021).

Krupnikov and Levine (2014) define external validity as the generalisability of findings to new environments and situations. Thus, if a study was performed under similar natural conditions, aided by triangulation, comparable results can be expected. Furthermore, this study addressed threats to external validity as proposed by Shadish and Myers (2001). That is, an *extended description of this study* was presented; there were *multiple interactions between the teacher and the researcher;* and lastly, at the end of this chapter *the researcher declares any biases* held towards this study.

The activities were informed by the pre-calculus curriculum and textbook, since it took place in a natural setting. The exercises in the activities were sequenced using the DCAT (Beggs & Mouw, 1980). This means that the exercises were organised in three groups of increasing complexity, i.e., easy, medium, and difficult. The easy exercises related to the DCAT's *Basic Cognitive Abilities,* referred to as DCAT 1; the medium exercises related to *Application Abilities*, referred to as DCAT 2; and the difficult exercises related to *Critical Thinking Abilities*, referred to as DCAT 3. In their investigation of the cognition levels of agricultural students, Torres and Cano (1995) designed a table that related the DCAT to Bloom's taxonomy. DCAT 1 was related to *Knowledge & Comprehension* in Bloom's taxonomy, DCAT 2 to *Application*, and DCAT 3 to *Analysis & Synthesis*. Thus, the DCATs are categorisations of the exercises from the activities, according to students' cognitive levels: low, medium, and high.

For the activity questions on the simplification of trigonometric expressions under *DCAT 1,* ten questions were discussed by the students over four days. After this, the students were required to complete an assessment. Similarly, under *DCAT 2*, ten questions were discussed by the students, on the same topic (the simplification of trigonometric expressions), also over four days; again, this was followed by an assessment. Thereafter, under *DCAT 3,* eight questions on the same topic as before were discussed by the students, in four days; this was followed by an assessment containing five questions on the simplification of trigonometric expressions at the same DCAT level. Only eight questions under *DCAT 3* were discussed because of time constraints and difficulty level. Correspondingly, the assessment on *DCAT 3* contained only five questions. All assessments were 45 minutes long.

3.3.2 Activity questions on proving trigonometric identities

Like the simplification of trigonometric expressions, apart from the DCAT levels, the selections and source of the activity questions were informed by the curriculum. The source of the questions was the prescribed textbook. Also, for the activity questions on proving trigonometric identities under DCAT 1, 10 questions on proving the trigonometric identities were discussed by the students, in four days; this was followed by an assessment. This assessment contained seven questions on proving trigonometric identities at the DCAT 1 level. Similarly, under DCAT 2, 10 questions on proving trigonometric identities were discussed by the students over four days, again followed by an assessment on this topic. This assessment contained six questions at the DCAT 3 level. Under DCAT 3, eight questions on proving trigonometric identities were discussed by the students over four days, and subsequently an assessment was given to the students containing five questions on proving trigonometric identities, at the same DCAT level.

Each of the activities had a ‘concept’ component; that is, the teacher explained what the concept of the simplification of trigonometric expressions entailed, by means of an easy example. In addition, each activity had a ‘question-solving’ component; that is, the observed struggles the students went through in solving the rest of the activity questions. ‘Easy example’ means an illustration that would give students enough information to solve the question, although they would struggle with the rest of the activity questions. This strategy walks a fine line between what Kapur and Bielaczyc (2012) refer to as ‘productive failure’ (see Chapter One) and direct instruction. Direct instruction is a teacher-centred instructional approach, formulated in the 1960s by North American behavioural psychologists Siegfried Engelmann and his collaborators, that has many benefits (Luke, 2014b). Instructional leaders claim that direct instruction leads to a form of quality control, since all instruction is following a strictly scripted delivery system.

Wijnia, Loyens et al. (2014) reported that direct instruction – such as working out examples on the simplification of trigonometric expressions, and the proving of trigonometric identities – is beneficial to students, since these concepts are familiar to them. Furthermore, correct procedures and knowledge are provided through direct instruction (Wijnia et al., 2014), which eradicates any misconceptions. Additionally, without direct instruction, the mind’s working memory – where all conscious thinking occurs – becomes cluttered with methods such as trial and error, making it less likely for the subject to learn new methods and procedures for question solving (Kirschner et al. 2006). Thus, direct instruction also alleviates the overuse of

cognitive resources without any hope of success, by helping to construct correct, ready-to-use domain knowledge (Klahr & Nigam, 2004). However, direct instruction in collaboration with non-direct instruction – and, as in the case of this study, productive struggles – will have a greater impact than using either method alone (Karp & Voltz, 2000).
The activity questions were designed so that students would progressively develop their own strategies.

3.3.3 Assessments on simplification of trigonometric expressions

Three assessments, each 45 minutes long, were issued to the students. The three assessments were based on DCAT 1, 2 and 3 respectively. Each assessment established whether the productive struggles experienced by students, during the activities for the simplification of trigonometric expressions on the 'easy, medium and difficult' scale, were resolved or not. Although the DCAT 1 and DCAT 2 assessments each contained seven questions, the assessment for DCAT 3 only contained five questions, because of the time constraint requiring completion of difficult questions within 45 minutes.

Each assessment, 45 minutes long, was issued to the students after they had completed the exercises at each of the DCAT levels. A number was allocated to each student assessment. The names of the students associated with these numbers were stored in a secure, locked cabinet. All assessments were scanned and uploaded to a secure, cloud-based information storage site so that in the event of any breach, the names of the students would be protected. This procedure is in line with Stuckey's (2014) process of 'de-identifying data' to ensure anonymity.

3.3.4 Assessments for proving of trigonometric identities

The three assessments about proving trigonometric identities followed the same pattern as the assessments for the simplification of trigonometric expressions. However, in addition to assessing the students on three levels, the assessments for identities also evaluated whether a student recognises when to start proving an identity on the left-hand or right-hand side.

3.4 The pilot study

In this section this chapter explains what constitutes a pilot study, and what the purpose of the pilot study was in this study. The findings of the pilot are also explored.
The focus of the pilot was on the application of the research instrument activities to a similar group of students to the one that participated in the study.
The pilot study took place at a neighbouring high school that was teaching the same topic. Students that enrol in pre-calculus must maintain an average of 85% in their mathematics courses; thus the students in the pilot were academically ‘like’ those students at the school where the main study took place.

One of the NCTM’s (2014) productive beliefs about teaching is to provide students with appropriately challenging questions. In this regard the pilot tested whether the activity questions were at the correct level for the students to struggle, but be able to solve the questions. These activities were designed to investigate students’ resolve when facing difficulty with manageable questions that relate to the simplification of trigonometric expressions and proving of trigonometric identities. These controls enabled the researcher to create suitable activities for this study. For instance, the activity questions were selected from South Carolina Department of Education-approved high-school textbooks, and if necessary, modified to increase or decrease the difficulty level. Furthermore, the researcher decided to only use the *sine*, *cosine* and *tangent* trigonometric ratios.

3.4.1 Rationale

The pilot was used to test the appropriateness of the methodology of the study (including data collection and analyses, and the levels of difficulty of the activity question), research instruments, a ‘dry run’ of data collection set-ups, etc. In addition, the pilot study gave an indication of what scheduling conflicts might arise for interviews between the researcher and the focus groups, and for meetings with the participating teacher. Dikko (2016) reported that a pilot study can help with interview techniques by emphasising ambiguities and difficulties in the questions, recording the questioning time to see if it is reasonable, and ascertaining whether the questions would elicit adequate responses for the main study.

3.4.2 Significance of the pilot study

The focus of the pilot was the application of the research instrument activities to a similar group of students to the one that participated in the study. The study benefited from the pilot, since its results indicated that the methodology used was appropriate. Furthermore, the researcher could establish a solid approach in his data analysis and could also adjust the difficulty level of the activities for the main study.

Lowe (2019) concurred, saying that:

> "Another feasibility purpose may include testing the planned data collection instruments/methods for all types of data for quality and appropriateness. If a qualitative study is planned, does the interview guide or the training of the interviewer(s) produce the type and depth of data needed to answer the study question(s)?" (p.117)

For this study, the research instruments were the activity questions that relate to the simplification of trigonometric expressions and the proving of trigonometric identities.

3.4.3 Procedures of the pilot

This section explains how the research instruments at the neighbouring school were tested. Of course, the pilot study also tested the methodology. To request permission to conduct the pilot study, the researcher e-mailed the pre-calculus teacher at a neighbouring school. The school has the same demographic make-up as the school where the main study eventually took place, so the choice was appropriate. The researcher explained the purpose of the pilot through e-mail and then met with that teacher. She was very enthusiastic, since the activities learned in the pilot study would help her approach the topic differently. Permission was sought from the school's principal to observe the teacher where the pilot study was held. The principal reluctantly granted the request. At a meeting between the researcher and the teacher conducting the pilot study, the researcher explained to the teacher her role in teaching the exercises of the research instrument on simplifying trigonometric expressions and proving trigonometric identities. During the teaching of the exercises from the research instrument, the researcher acted as an observer.

3.4.4 Findings from the pilot

The main results of the pilot revealed that the level of difficulty and the number of questions in the activities had to be adapted. For example, since the pilot took place at a school that had

90-minute class periods, but the school where the main study took place had 50-minute class periods, the number of questions per class period had to be reduced. Also, the students in the pilot could easily solve the 'difficult' pilot activity questions. Thus, for the main study, the difficulty of the 'difficult' activity questions had to be increased. Next, at the beginning of the class the pilot study teacher allowed students too much time to engage in idle talk while she was busy at her computer. This relaxed atmosphere in the classroom allowed the students not to focus fully on the activities and to get distracted by personal issues. Moreover, the teacher did not allow enough time for the students to grapple with some of the questions before intervening with hints on how to solve the questions.

Thus, for the main study the researcher and the participating teacher agreed that classroom distractions should be kept to a minimum, and more attention should be placed on the activities. Furthermore, the results of the pilot study made the researcher aware of the busy and sometimes conflicting schedules of the students who would possibly participate in the focus group.

3.5 Research procedure

This section explains the procedure used to gather the data. It starts by describing the research site, and how the sampling occurred. Next, the research instruments – i.e. the activity questions – are presented. This section then explains how data was collected after the activity questions, as well as how the focus group and teacher meetings were conducted and recorded. Lastly, to show the elimination of any biases, the roles of the researcher and the teacher are clearly articulated.

3.5.1 Description of the research site

This study took place at a rural high school in South Carolina, United States of America, with a population of about 500 students. The high school's mission and belief statement is to create an environment where each student can be afforded a quality education, so that students will be lifelong learners, become responsible citizens, and be successful in the workplace. The high school curriculum includes subjects offered at the following levels: college preparatory, honours, Advance Placement (AP), dual credit with the neighbouring technical college, and Career Technical Education (CATE). The Advance Placement courses are administered by the College Board. The College Board is an American non-profit organisation that was formed in 1899 as the College Entrance Examination Board, to increase admission to tertiary education. The Career Technical Education level courses include mechatronics, introduction to health (nursing), accountancy, computer education and agriculture. The school has a strong farming component, with a thriving greenhouse and farming practices. There are 33 full-time teachers with 18 educational support staff members. Each classroom is fitted with all the necessary educational tools as requested by each teacher, as well as central air-conditioning and heating.

The school is situated in a small town with a population of 1420, according to the 2017 estimates from the US Census Bureau, that has shown a steady decline over the past eight years. About 34% of the town population holds a high-school diploma, 23% did not complete high school, about 17% have some college education, 7% hold an associate degree, about 13% hold a bachelor's degree, and 6% have a postgraduate degree. However, the school population of about 500 students consists of students from two neighbouring towns as well.

3.5.2 Research sample

Shaheen and Pradhan (2019) contended that sampling procedures in qualitative research are not well defined. Moreover, they argued that the selection of the participants in a qualitative research study depends on the purpose of the study, as well as the discretion of the researcher. Nonetheless, the researcher chose *purposeful sampling.* Shaheen, Gupta and Kumar (2016) gave reasons that purposeful sampling should include information-rich samples to give an in-depth view of the study. For this study, since the productive struggles during the simplification of trigonometric expressions and the proving of trigonometric identities were observed, the participating school's pre-calculus class seemed the appropriate choice as a sample. All the students in the pre-calculus course participated in the study.

All the students in this study came from a middle-class background and were taking college preparatory classes at the high school. Their ages ranged from 16 to 17 years. Five boys and ten girls participated in this study, and all the students were English first-language speakers.

3.5.3 Data collection techniques

This section explains the approach used to collect data for this study. Three types of data were collected, namely the video recordings of the classroom activities, the assessment scripts after the students' assessments, and the audio recordings of the focus groups.

3.5.3.1 Data from video-recorded learning activities

The main sources of data were recorded videos of the student-student and student-teacher interactions, to observe the productive struggles the students encountered while simplifying trigonometric expressions and proving trigonometric identities. Consequently, the activities in this study were videotaped, and importantly, took place in real time in a normal classroom used for instruction at the participating school. The use of real-time recordings is in line with Noble and Smith's (2014) notion that qualitative research should take place in its natural environment, making it easy for the observer to discern which events lead to which consequences. Video recordings were made to SD cards, which were removed from the camera after each recorded set of activity questions.

3.5.3.2 Data from students' written assessments

This section explains the use of the data that was collected during the study.
While the assessments compared student performances in simplifying trigonometric expressions and proving trigonometric identities, they were also used to verify whether the students' productive struggles during the learning activities were resolved or not. Similarly, the students' written assessments were analysed, to investigate why certain productive struggles persisted when the students simplified trigonometric expressions and proved trigonometric identities. Photocopies of the students' written work were made on which to make notes on the work, and both originals and copies were secured in a locked cabinet. This last is a requirement of the data management policy of the University of Cape Town.

3.5.3.3 Data from students' focus group interviews

After each assessment, the students' scripts were marked by the researcher. A small group of five students participated in focus group interviews. To get a fair representation of the students in the class, two low ability, two medium ability and one high ability student was approached by the researcher to participate in the focus group. All five students agreed but warned that some of them will have afterschool activities and lunch break duties that they might have to attend to.

This group was handed back their written scripts and given an opportunity to explain their solutions to the researcher. The main purpose here was to obtain evidence as to whether the students' productive struggles that were observed during the learning activities were resolved during the assessments. Focus group interviews gave the researcher a deep understanding of the nature of the errors the students committed in simplifying trigonometric expressions and proving trigonometric identities.

The recorded focus group interviews were uploaded to a secure cloud-based site and then transcribed verbatim. The researcher created a calm and inviting atmosphere by assuring the group that the information gathered during the interview was strictly for research purposes, and that it would not be used to judge them in any way. Each recorded interview session was started by the researcher stating the date and the aim of the interview. The interviews were a mix of informal, conversational interview style and standardised open interview style (Turner III, 2010). For the interviews sessions the researcher used a pre-determined interview guide (which appears in Appendix 18), though follow-up questions were also asked.

3.5.3.4 Researcher and teacher meetings

Pre-teaching meeting

The aim of the pre-teaching meeting was for the researcher to outline the sequence of events expected of the teacher when facilitating the learning activities. The researcher explained to the teacher how the activity questions were sequenced, and in what order they should be tackled by the students. The researcher also stressed to the teacher the importance of not giving the students the solution to an activity question, but rather guiding them to it. The researcher was responsible for setting up the video cameras before the students began to attempt the activity questions. Since all the classrooms at the school were adequately equipped with all required teaching materials, there was no shortage of resources. Additionally, the researcher made the teacher aware of the type of evidence needed for this study – students struggling productively when they simplified trigonometric expressions and proving trigonometric identities. The researcher explained to the teacher that the video recordings should start after the teacher took attendance and the students had settled down and were ready to start with the activities. In this way, little time was wasted. The video recordings stopped at the end of the class period. Six recorded pre-teaching meetings took place between the researcher and the teacher. These meetings happened after the different DCAT activities.

Post-teaching meetings

The purpose of the post-teaching meetings was for the researcher and teacher to reflect on the learning activity facilitation performed by the teacher. This reflection included discussion of methods to best observe the productive struggles experienced by the students during the simplification of trigonometric expressions and proving trigonometric identities. Once this study began, the researcher met with the participating teacher two or three times per week before school or at lunch, when they had time to discuss how the study was progressing. Six recorded and four short non-recorded meetings took place between the researcher and the teacher. The short non-recorded meetings took place primarily to clarify the teacher's role in the classroom.

3.5.4 Role of the teacher

Primarily, the teacher's role was to facilitate the activity questions and support the students when they struggled in simplifying trigonometric expressions and proving trigonometric identities. In addition, the teacher's role included the pre- and post-activity meeting obligations discussed in the previous sections. The teacher also gave the researcher valuable information about classroom dynamics that was not captured in the video recordings.

3.5.5 Role of the researcher

The roles of the researcher in this study were as follows: firstly, the researcher sought permission from the relevant stakeholders – that is, the district superintendent, the school principal, the participating teacher, the participating students, and the parents or legal guardians of the participating students. The researcher also sought and obtained an ethics clearance certificate from the University of Cape Town. Secondly, the researcher performed a pilot study at a neighbouring school to ascertain the feasibility of performing the main study at the chosen school. The researcher was responsible for designing the learning activities for the pilot study and administering the pilot. The researcher was also responsible for collecting the relevant information from the teacher who performed the pilot study. Thirdly, the researcher designed learning activities to investigate the students' productive struggles when simplifying trigonometric expressions and proving trigonometric identities. As in the pilot, the researcher was responsible for the collection and safekeeping of the video SD cards, and of the students' written work and assessments. The researcher uploaded the video recordings and the students' written work and assessments to a secure, cloud-based data-storage site. Fourthly, the researcher conducted focus-group interviews and teacher meetings, and uploaded all the recordings to the same cloud-based site. Lastly, the researcher was responsible for analysing the data.

3.6 Data analysis

This study uses qualitative data analysis approaches supported by descriptive statistics. This section discusses the data analysis approaches used to analyse the data collected from recorded learning activities to do with simplifying trigonometric expressions and proving trigonometric identities. Then there is a discussion of how data from assessments and focus group interviews were analysed using qualitative research approaches.

3.6.1 Qualitative data analysis (*how the data was analysed*)

This subsection discusses the analysis of the video recordings and focus group interviews. This study used an open-ended approach (Cohen, Manion and Morrison, 2007) to analyse the data. This means that the data was analysed in the context of the research questions as well as of any theories emerging from the data itself. After watching the videos of the classroom interactions (both student-student and teacher-student) and listening to the interviews on the audio tapes, any patterns that emerged were noted and categorised. This process was iterative, as the same videos were watched several times to make sure the categories and patterns were identified accurately (Braun & Clarke, 2006).

3.6.1.1 Analysis of data from video-recorded activities

Analysis of video recordings

This section discusses the study's data collection method for classroom observation. Video recordings were used as a means of data collection for classroom observation. Evidence (informed by the literature) was produced to demonstrate the advantages of using video recordings in this study. The analysis of the video recordings took place after the recordings were uploaded to a secure cloud-based site.

During any interaction between student and student in a classroom setting, students talk, gaze, gesture, and use facial expressions, movement, and material objects to exchange mathematical ideas. All this information can be captured by video (Goodwin, 2013). To capture and make sense of the intricacies and dynamism of the knowledge in use between students, Hall and Stevens (2016) contended that these actions should be captured in a 'motionless' way. This means that all video recordings of the learning activities were transcribed verbatim.

However, Ramey et al. (2016) argued that transcription can be subjective; that is, the researcher transcribes according to his agenda and point of view. To counter this, Ramey et al. (2016) suggested creating multiple transcriptions that would help researchers to capture interactions between students. This idea is based on the recursive notion proposed by Goodwin (2013), in which researchers come up with a theory, then view and re-view the video, then document the interactions of the students as seen from different perspectives. This study complied with these suggestions by Goodwin (2013) and Ramey (2016) by employing a second transcriber.

Video offers a unique opportunity for the researcher to pay attention to other modalities in addition to speech. Derry et al. (2010) and Goodwin (2013) suggested that it is important to be sensitive and intentional about choosing a semiotic field for transcription. Recursive transcriptions of multiple semiotic fields give guidance for noticing during analysis of the transcripts (Ramey et al., 2016). Stevens (2012) suggested scrutinising these semiotic fields: S*tudents' signalling and pointing* might indicate something that is not immediately obvious, but which might be of interest to the researcher. *Stare and attention* might indicate disinterest in the work, or other unrelated questions that the student might have. *Body position*, such as standing with hands on hips or *moving* up and down might indicate frustration or confusion. *Touching* the face or rubbing the hand over the face might have some implicit meaning that the researcher should be aware of. The lowering of the student's voice might indicate a shyness to ask for help, or a fear of retribution from fellow classmates for giving an incorrect answer; thus, *tone and subtle nuances* are another semiotic field that deserves attention. When students fidget, as in engagement *with material objects,* it might be a call for attention from the teacher or fellow classmates, and *facial expression* may give a clue about confusion or surprise. Thus, to get a multimodal description of productive struggles in simplifying trigonometric expressions and proving trigonometric identities, multiple transcriptional passes were made, with each pass focusing on a different semiotic field.

To address the question of *how* to transcribe the different semiotic fields, Ramey et al. (2016) suggested *visual transcription.* This entails making multiple representations of a scenario, in this case to evoke the struggle that the student was encountering. For example, during an excerpt of a productive struggle, the first transcription pass would focus only on facial expressions; the next pass would focus on tone and subtle nuances, and so on. Together, the different transcriptions would give a holistic view of the struggle.

The influence of teacher questioning and noticing on the semiotic fields was also noted. This means a note was made of how the teacher responded to *stare and attention, body position, touching, tone and subtle nuances, engagement with material objects, and facial expressions.* Since the teacher was not a subject of the research, the latter were only used to enrich the interpretations of the students' productive struggles during the simplification of trigonometric expressions and proving trigonometric identities.

3.6.1.2 Documentary analyses from students' written assessments

In this section the errors made by the students in written assessments were analysed using the NEA framework. The purpose of this analysis was to provide evidence on the nature of the errors committed by students in the assessments and indicates whether any productive struggles from the learning activities still prevailed in the assessments. Also, this study could ascertain whether there was a difference in the students' performance in the assessments on the simplification of trigonometric expressions and the proving of trigonometric identities. Fifteen assessments were analysed. Each student's script was scored by the researcher based on the types of errors made by the students. These errors were classified (based on NEA) as errors made under the *reading, comprehension, transformation, process skill,* or *encoding hierarchy* categorisations.

3.6.1.3 Students' focus group interviews

In general, the purpose of a focus group is to arrive at some nuanced explanations about students' feelings and misconceptions, and to get a deeper understanding of the types of errors committed by students during the assessments, as well as how these errors relate to the students' productive struggles that were observed during the learning activities. The focus group interviews after the assessment could confirm whether the learning issues raised by students during the productive struggles had been resolved or not.

The NEA framework was used to categorise and analyse the errors. The analysis of interviews and video recordings revolves around "immersing oneself within the data to gain detailed insights into the phenomena being explored; developing a data coding system; and linking codes or units of data to form overarching themes/concepts, which may lead to the development of theory" (Noble & Smith, 2014, p.3). The probing question was: "Which question(s) on the assessment gave you difficulties in solving?" This approach aligns with Kitzinger's (1997) notion that a focus group should elicit students' explanations for why they struggled with difficult questions.

Ranney et al. (2015) reasoned that in-depth interviews with a focus group would give insight into issues that might not otherwise have been detected. Regarding this study, the focus group interviews thus helped the researcher to better understand issues that arose during observation of videos or analysis of student interactions in the classroom. McCormick and James (2018) contended that the integrity of an interview may be compromised by delay; thus, interviews took place on the same day as the activities, or at the latest the following day.

The focus group met in a classroom, and the interviews were recorded on a voice recorder. This machine provided clear recordings that were ideal for transcriptions; this was in line with Stuckey (2014), who suggested that it was best to use tape recorders when recording focus group interviews, as this would eliminate the distraction of taking detailed notes. Thematic analysis uses discernible themes of human behaviour (Sharma, 2013). After listening to the recordings numerous times and reading the transcripts, common patterns were noted that emerged either from paraphrasing or from direct quotes.

Also, when the researcher observed certain themes during the interview process, he would ask follow-up questions, in the format: “I notice that *x* appeared, is that correct?” This gave him feedback, reaffirming that certain patterns were evolving. In addition, this question and ‘requestioning’ allowed clarification of tone and inflection. This process could have been done after the transcription of the interviews, as suggested by Aronson (1995), but that would have been impractical in this study, since by then the students could have forgotten the context of the interview questions. Lastly, the related literature was studied, to validate the themes and storyline that were emerging.

3.7 Reliability, credibility, dependability and validity of this study

In this section, the concepts of the reliability, credibility, dependability and validity of this study will be explored.

3.7.1 Reliability, credibility, dependability and validity in qualitative research

Rose and Johnson (2020) defined r*eliability* in qualitative research studies as the soundness of a study in relation to the appropriate methods chosen and how those methods are implemented in the study. For this study, to investigate the types of productive struggles experienced by a group of students when they simplified trigonometric expressions and proved trigonometric identities, well-designed research instruments relating to the simplification of trigonometric expressions and proving of trigonometric identities were issued to the participating students. Both the research instruments and the research methodologies were piloted at a neighbouring school. Accordingly, the research instruments were adapted based on the results from the pilot. For example, the difficulty level of the research instruments and the time allowed to complete them was adjusted for the main study. Reliability in qualitative study also seeks to evaluate its quality; in this study, this is assured by the study’s credibility and dependability.

Credibility in this study was assured by data triangulation, researcher reflexivity and relatively long (six months) period of time over which the study took place. Triangulation in this study was accomplished by corroborating video recordings and assessment analysis with the focus group interviews. Also, a qualitative study should be able to increase its transparency by exposing any shortcomings and decreasing researcher bias (Jefrydin et al., 2019). In Chapter Six, the researcher declared all known limitations to this study. One major shortcoming of this study was the population size. The researcher also declared any possible biases.

Lincoln et al. (1985, p.290) eloquently described the issue of credibility: "How can an inquirer persuade his or her audiences that the research findings of an inquiry are worth paying attention to?" This study mirrors the collection methods used in the qualitative research literature; that is, activity questions, observation by video, teacher meetings, focus group interviews, and analysis of student work, with a relatively small sample size. In this study, video was used to record and then examine the real-time struggles that the students experienced in simplifying trigonometric expressions and proving trigonometric identities. These struggles were noted and explored during the focus group interviews. Moreover, as these mathematical topics form part of the pre-calculus syllabus, they were taught by a regular teacher in a classroom of normal instruction. This is in line with Noble and Smith's (2014, p.1) recommendation that qualitative research should seek to understand a phenomenon in its natural setting, and to "understand phenomena from the perspective of the individual or group". Furthermore, all themes that emanated from the teacher meetings and focus group interviews were clearly defined and consistent. This consistency of codes and themes is what Miles, Huberman and Saldana (2014) regarded as a technique to ensure reliability in a qualitative study.

Dependability in a qualitative study considers whether the collection of data was sound (L.Haven & Van Grootel, 2019). The data for this study was collected through video recordings and the assessment scripts of the students.

Validity refers to the process of establishing the accuracy of the findings from the perspectives of the researcher, the participants or the consumers of the research (Lincoln, Lynham & Guba, 2013). Leung (2015) described validity in qualitative research as the suitability of certain tools, data and procedures in performing an inquiry. In the context of this study, the question was: was it suitable to use video and interview analysis, supported by students' written work and assessments, to answer the research questions? Moreover, were the research questions appropriate for investigating productive struggles? The sections that follow respond to these questions, confirming the validity of this study. One way of testing the inferences made by a researcher is to ask interviewees if they agree with the conclusions drawn from the interviews. This was done multiple times during the focus group interviews. Singh (2014) proposed replacing validity and reliability in qualitative research with the concept of trustworthiness, rigour and quality, which Johnson (1997, p.282) contended is "defensible", in the sense that the researcher can use triangulation, prolonged contact with the subjects under investigation, peer review and reflexivity to establish confidence in their findings (Lincoln et al., 1985).

3.7.2 Minimising of research bias

Other than in the teacher meetings and focus group interviews, the researcher played a non-participatory role in this study. While the learning was being carried out, the researcher did not interfere with the teacher's facilitation, though he was present in the classroom. The video cameras were in an inconspicuous position in the classroom so as not to interfere during the lesson activities. For this study, two people worked on the transcriptions; one created the transcriptions and the other verified that the data in the transcriptions had been captured correctly. This method minimised any subjectivity and bias that may have arisen during the transcription process.

3.8 Ethical issues

Central to any social research is the participants' anonymity and confidentiality. By using pseudonyms for the participants and their location, researchers can assure participants that no identifiable information in journals, reports or presentation can be traced back to them (Şendil & Sönmez, 2019). Confidentiality and anonymity are closely related, but in ethics literature, Şendil and Sönmez (2019) observed, confidentiality is like privacy. In addition, participant consent is essential in the research field (Connelly, 2014). Having been involved with high-school students for over 25 years, the researcher in this study agrees with Fouka and Mantzorou (2011), who contended that particular care should be taken when children are involved. In addition, consent should be sought from the parents, as encouraged by the

American Educational Research Association (2011). As explained, this study adhered to all these recommendations.

The sequence of events for acquiring permission to conduct this study was specific and deliberate. At the start of study registration at the University of Cape Town, ethics clearance was sought from the Faculty of Science Research Ethics Committee and was granted (reference number FSREC 21-2019; see Appendix 5). Before the start of the study, permission was sought through issuing a universal permission letter (Appendix 1-4) to the participating stakeholders, adapted according to their positions in the hierarchical educational structure – from the top down, district superintendent, school headmaster, teacher, parents and student. First, permission was sought from the district superintendent. After permission had been granted, the headmaster was approached for permission to perform this study at the school. Thereafter, permission from the participating teacher was sought and received, and lastly from the students and their parents (see Appendices 1-4). All confidential video and audio recordings were captured and stored on an encrypted storage device and then on a cloud-based service. The storage device and cloud-based service were password protected.

3.9 Summary

Chapter Three laid the foundation for Chapter Four, which analyses the data collected from the study on simplification of trigonometric expressions in detail. It briefly discussed the historical evolution of qualitative research methodologies and presented a motivation for supplementing this methodology with descriptive statistics. The reliability and validity of qualitative research methodologies in the context of this study were discussed.

This chapter described the main data collection and data analysis approaches used in this study – these included video observations and analysis, focus group interviews, and documentary analyses using the NEA framework of students' written assessments. Using evidence from the literature, this chapter showed that for this study, video analysis, focus group interviews and student assessments were appropriate research tools.

Lastly, since this study involved minors, ethical clearance was sought and received from the University of Cape Town and the students' parents or legal guardians.

CHAPTER FOUR: DATA ANALYSIS – SIMPLIFICATION OF TRIGONOMETRIC EXPRESSIONS

4.1 Introduction

In this chapter, this study analyses the data captured when a sample of high school students simplified trigonometric expressions at DCAT levels 1, 2 and 3. While there are elements of ATD in the analyses, overall the productive struggle framework was used. Throughout the analyses, this study uses the word *activities* and *activity questions* interchangeably.

4.2 Simplification of trigonometric expressions

This study focused on high-school students' productive struggles during two types of activities: the simplification of trigonometric expressions, and the proving of trigonometric identities. This chapter shows the analysis of the video-recorded activities, assessments, NEA and focus group interviews when the students simplified trigonometric expressions. A focus group interview took place after each DCAT assessment and was used to seek clarity on the participants' understanding of the NEA approaches to the simplification of trigonometric expressions. Analysis of data from DCAT 1, 2 and 3 levels were performed. Each DCAT data level was made up of two activities: Activity 1 and Activity 2 questions. Section 4.2.1 of this chapter analyses DCAT 1 data. Section 4.2.2 analyses DCAT 2 data; and Section 4.2.3 analyses DCAT 3 data.

Each set of Activity questions at the different DCAT levels is analysed as follows. First, the distribution of the productive struggles by the students for Activity questions is presented in a table. Next, the question for which the students experienced the most productive struggles is analysed according to the productive struggle framework created by Warshauer (2014). Although other studies from the literature are also cited, this study used Boaler and Brodie's (2004) nine question types (discussed in Chapter Two) when the teacher questioned a student during their struggles. Also, to analyse how the teacher responded to a student's struggle, this study used Warshauer's (2014) *teacher response to productive struggle* (discussed in Chapter Two).

The analysis of the rest of the questions in the Activity is presented in a table. This is followed by the NEA of the first assessment. The NEA will reveal if all the productive struggles from the activities were resolved or not.

4.2.1 Data analyses from DCAT 1 – Basic cognitive abilities

The data analysis in this chapter starts with examples of the types of productive struggles that the participating students encountered when they simplified trigonometric expressions in the DCAT 1 activities. This study explored examples of students' productive struggle types and the errors they made at DCAT 1 level by analysing their actions for the simplification of trigonometric expressions at the basic cognitive ability level. Data was collected through video recordings of Activities 1 and 2, and from the students' assessment scripts at the end of the DCAT 1 level. In addition, audio recordings were made of the focus group interviews. These interviews gave the researcher a deep understanding of the nature of the errors that students committed regarding simplification of trigonometric expressions. Again, the focus group interviews after the assessment confirmed whether the learning issues raised by students during their productive struggles had been resolved or not.

All recorded data were transcribed verbatim. All transcriptions were analysed using Atlas.ti software. NEA was performed to categorise the types of errors students made when simplifying trigonometric expressions. The next section discusses the data analysis of the first activity questions at DCAT 1 level.

4.2.1.1 Distribution of students' productive struggles on DCAT 1 Activity 1 questions

In this section, this study analyses activity 1 with questions at DCAT 1 level. The questions are summarised in Table 4.1 below, which presents findings about the productive struggles observed when the students simplified a trigonometric expression (shown in the left-most column of the table). The analysis was completed by observing and analysing the video recordings of the questions answered by the students in the classroom in real time.

Table 4.1: Distribution of students' productive struggles from DCAT 1, Activity 1 questions

Questions	Types of productive struggle					
				Expressing misconceptions and errors		
Simplify each of the following trigonometric expressions	Getting started	Carrying out a process	Uncertainty in explaining and sense making	Misconceptions	Errors	Total
$sinxtanxcosx$	3	3	1	0	0	7
$\frac{cosx}{tanx}sinx$	1	1	1	1	1	5
$\frac{sinx + cosx}{sinxcosx}$	1	1	1	1	0	4
Totals	**5**	**5**	**3**	**2**	**1**	**16**

In the next section this study analyses the activity question(s) for which the students experienced the most productive struggles. The DCAT 1 activity has more than three questions (see Appendix 6), but only questions for which students experienced productive struggles are shown in Table 4.1. The number in each cell indicates the number of times a specific type of productive struggle was observed from the video recordings. The students exhibited the most examples of productive struggles when they simplified the trigonometric expression $sinxtanxcosx$. The 'Total' column on the right side of the table tallies the number of examples of students' productive struggles that were observed *for each question*. The totals at the bottom of the table represent the frequencies of a type of productive struggle across the questions.

From the questions not included in the Table 4.1, the teacher wrote the following trigonometric identities on the top right-hand corner of the whiteboard:

$$sin^2x + cos^2x = 1 \text{ and } tanx = \frac{sinx}{cosx}.$$

The teacher then guided the students through the first question; that is, to simplify the trigonometric expression $sinx\frac{1}{tanx}$. The teacher started this trigonometric expression by asking the students what the most appropriate trigonometric identity substitution would be to perform the simplification. After the simplification of $sinx\frac{1}{tanx}$ the students continued independently, easily solving the second question, which was the simplification of the trigonometric expression $sin^2x + cos^2x + 1$. Some of the students exclaimed, "That's easy", and "I like that". Thus, no productive struggle was observed for the simplification of $sinx\frac{1}{tanx}$ and the simplification of $sin^2x + cos^2x + 1$, which were thus excluded from the table. Next, the particular productive struggles observed in the question with the highest number of struggles are presented.

4.2.1.2 An analysis of the question with the highest number of productive struggles on a question from DCAT 1 Activity 1

Although the students exhibited the greatest number of productive struggles when they attempted to simplify the trigonometric expression $sinxtanxcosx$, they did not exhibit the 'expressing misconceptions' and 'errors' types of productive struggle. In contrast, the simplification of the trigonometric expression $\frac{cosx}{tanx}sinx$ exhibited two fewer productive struggles than the simplification of $sinxtanxcosx$, but in this case all the different productive struggle types were present. Thus, the simplification of the trigonometric expression $\frac{cosx}{tanx}sinx$ was chosen for analysis. After the teacher explained to the class how to simplify the trigonometric expressions $sinx\frac{1}{tanx}$ and $sin^2x + cos^2x + 1$, one of the trigonometric expressions he asked the class to attempt to simplify was $\frac{cosx}{tanx}sinx$.

This study begins the analysis of each productive struggle with an excerpt from the incident, and then continues with a brief explanation of what transpired in the classroom. Throughout Chapter Four, T = teacher, S = struggling student and C = one or more classmates. The researcher's comments are in italics and square brackets.

<u>Getting started</u>

S: $\frac{cosx}{sinx}.sinx$? Hmm… I think I might have messed up, because I put that as an equal sign. *[she had written " $\frac{cosx}{tanx} = sinx$ " on her paper]*

C: Where did you get $\frac{cosx}{sinx}$?

S: That? I multiplied them *[indicating with her thumb and pinky between tanx and sinx, meaning cosx=sinxtanx, so* $\frac{cosx}{sinx} = tanx$. *She then rewrote* $\frac{cosx}{tanx}.sinx$ *on the board]*

C: Oh…

S: Would you put this over one? *[meaning* $\frac{cosx}{tanx}.\frac{sinx}{1}$*]*

T: That is fine.

S: Then [do] you just reciprocal or what?

T: What do your classmates say?

Here, a student who was struggling to 'get started' with the simplification of the expression $\frac{cosx}{tanx}sinx$ went to the whiteboard and attempted to start simplifying the trigonometric expression. After explaining to the class how she/he had (erroneously) attempted to solve $\frac{cosx}{tanx}.sinx$, the student wanted to know from the teacher if she could "just reciprocal or what?". The teacher *responded* by saying:" What do your classmates say?" This statement is prefaced by the teacher making a *general observation* (Teuscher et al., 2017). A *general observation* implies that no specific mathematical observation of the students was made. Linked to this general observation by the teacher is *general interpreting.* General interpreting implies that no specific mathematical interpretation was made by the teacher. Warshauer (2014) refers to this type of response by the teacher as a *vague* or *unfocused* response. By allowing the class to respond to the struggling student, the teacher avoided lowering the *cognitive demand* of the question by *telling* the student the answer or by *guiding* the student to an answer without the necessary connection and meaning, as suggested by Jackson, Gibbons and Sharpe (2017). This type of questioning where the teacher elicits help from the class is called a *general discussing* question (see Chapter Two for question types). Also, in this instance the student lacked *strategic competence.* That is, the student could not formulate the problem which Kilpatrick et al. (2001) describes as one of the characteristics of strategic competence. Furthermore, this study agrees with Wilson and Heid (2011), who contended that strategic competence requires procedural fluency and conceptual understanding. In this excerpt the student lacked both conceptual understanding and procedural fluency; the student was unsure whether to write $sinx$ as $\frac{sinx}{1}$. In terms of TRU (see Chapter Two), the cognitive demand of the activities was at DCAT 1 level. The reader is reminded that Schoenfeld (2018) contended that cognitive demand refers to the extent to which students have opportunities to grapple with and comprehend disciplinary ideas. Since the activity questions were at DCAT 1 level, there was ample opportunity for the students to grapple with the activity questions. There was no issue regarding the students' *agency*, since the whole class participated in the activity question. Equitable access to content was addressed by inviting *any* struggling students to write their answers on the board.

<u>Carrying out a process</u>

S: $\frac{cosx}{\frac{sinx}{cosx}} sinx$

C: You can cancel out the sines.

S: You don't do the reciprocal? Don't you do $\frac{sinx}{cosx}$?

S: Don't you do $\frac{sinx}{cosx}$?

C: No! You multiply *cosx* over one with *cosx* over *sinx.*

S: $\frac{cosx}{1} \frac{cosx}{sinx}$ = cos…

C: No! Times *sinx* over 1!

S: $\frac{cosx}{1} \frac{cosx}{sinx} . \frac{sinx}{1} = \frac{cos^2x}{sinx} \frac{sinx}{1} = cos^2x$

T: Do you understand where that is coming from?

The algebraic process $\frac{cosx}{\frac{sinx}{cosx}} sinx = \frac{\frac{cosx}{1}}{\frac{sinx}{cosx}} . \frac{sinx}{1} = \frac{cosx}{1} \frac{cosx}{sinx} \frac{sinx}{1} = cos^2x$ seemed to evade the struggling student. An interaction between the struggling student and the class ensued – with help from the class, and minimal input from the teacher – to unravel the struggle the student was experiencing. Unlike the study conducted by Rankweteke (2020), who investigated cooperative learning for conceptual understanding of trigonometry, this study found that students interacted eagerly to resolve any productive struggle. Furthermore, the timing of the teacher questioning aligns with what Martino and Maher (1994, p.4) posited in Chapter Two:

> "[A]fter students have built a solution, consulted with each other and proposed a solution that they believe is valid, they are ready for the challenge to justify and/or

generalise their solutions. It is at this time that the teacher's role of interaction with students becomes critical."

This type of questioning, where the teacher tries to focus the attention of the struggling student on essential elements to problem solve, is called using *orienting and focusing* questions. Warshauer (2014) classifies this type of teacher response as *probing guidance.* That means the teacher is promoting student self-reflection and bases his guidance on student thinking. Also, in this excerpt, the student lacks *conceptual understanding, procedural fluency* and *strategic competence*; the student is unsure about the process of multiplying with the reciprocal of a fraction. The student is also unsure about the process of multiplying two fractions, and lacks the creativity to see that $\frac{cosx}{\frac{sinx}{cosx}} sinx = \frac{\frac{cosx}{1}}{\frac{sinx}{cosx}} sinx.$

Uncertainty in explaining and sense-making

S: $\frac{cosx}{tanx} sinx = \frac{cosx}{\frac{sinx}{cosx}} sinx.$

What we did was to cancel the $sinx's$ and were left with $\frac{cosx}{\frac{1}{cosx}}$, then we did cosine over cosine. But would it not be cosine divided by one over cosine?

T: Yeah…

S: *[turns towards his classmates and nods his head, admitting he was wrong].*

We just did equals 1.

$\frac{cosx}{\frac{1}{cosx}}$ …I don't know what to do there.

T: Okay. Um... but really, it is *cosine* divided by one over *cosine*, which is not wrong. But how do we go from here: *[*$\frac{cosx}{\frac{1}{cosx}}$*]* to here: cos^2x?

C: You multiply cosine by cosine over one.

The student in the excerpt above experienced uncertainty about his solution, saying they had "just cancelled out the $sinx's$ and were left with $cosine$ divided by one over $cosine$." However, he thought it was wrong; he continued by saying: "I don't know what to do there."

In this instance the teacher's questioning was specific, to elicit a precise response, as per Menezes et al. (2014). The teacher asked: "… But how do we go from here: $\left[\frac{cosx}{\frac{1}{cosx}}\right]$ to here: cos^2x?", so as not to show a bias toward either answer. The class responded by saying: "You multiply $cosine$ over one by $cosine$ over one." By his question, the teacher helped to resolve the productive struggle. This type of questioning, where the teacher tries to show the link between two answers, is called a *linking* and *applying* question. In this instance, the teacher responded by guiding the students based on their thoughts, which Warshauer (2014) refers to as *probing guidance.* In this excerpt the student exhibited a lack of adaptive reasoning. This study agrees with Alexander, White and Daugherty (1997) that the lack of *adaptive reasoning* stems from a shortage of base knowledge. In this case, the student did not understand the mathematical processes involved when dividing two fractions.

This excerpt explains the important interplay between the knowledge block and the practical block of the ATD. The teacher used the technology – that is the simplification of a fraction divided by another fraction – to justify the technique applied to simplify the expression.

<u>Expressing misconceptions and errors</u>

C: Where did you get $\frac{cosx}{sinx}$?

S: That? I had multiplied them *[indicating with thumb and pinky a link between tanx and sinx, meaning cosx = sinxtanx, so* $\frac{cosx}{sinx} = tanx$*. The student then rewrites* $\frac{cosx}{tanx}.sinx$*].*

S: Would you put this over 1? *[meaning* $\frac{cosx}{tanx}.\frac{sinx}{1}$*]*

T: That's fine.

S: Then… you just reciprocal, or what?

T: What do your classmates say?

C: I don't know if that is correct, but I replaced $tanx$ with $\frac{sinx}{cosx}$.

T: That's okay.

In this excerpt, the struggling student made an error by setting $\frac{cosx}{tanx} = sinx$. This action was followed by intervention from the teacher to allow the class to resolve the error. Of importance here is the teacher's response to the student's struggle. By showing *no clear* *connection* between his response and the student's struggle, the teacher places the onus of analysing the struggling student's difficulty on the class. Questioning by the teacher that elicits help from the class is called *generating discussion* questions. The responses by the teacher, such as "That's fine" and "That's okay" is labelled as *vague* or *unfocused.* Similar to the excerpt on 'uncertainty in explaining and sense-making', the lack of prior knowledge (knowing that $tanx = \frac{sinx}{cosx}$) causes the student not to think logically about their work. Hence, the student lacked *adaptive reasoning.*

4.2.1.3 Summary of the students' productive struggles from DCAT 1 Activity 1 questions

Table 4.2 below summarises the examples of students' productive struggles and the evidence to support these struggles seen during DCAT 1 Activity 1 questions.

The entries under each different type of productive struggle contain excerpts from the session where the teacher or the class tried to resolve the productive struggles relating to the questions on the right-hand side of the table.

Table 4.2: Summary of the examples of students' productive struggles from DCAT 1 Activity 1 questions

Questions	Types of productive struggle				
				Express misconceptions and errors	
Simplify each of these expressions	**Getting started**	**Carrying out a process**	**Uncertainty in explaining and sense making**	**Expressing misconceptions**	**Errors**
$sinx.\frac{1}{tanx}$	Teacher illustrated	Teacher illustrated	Teacher illustrated	Teacher illustrated	Teacher illustrated
$sin^2x + cos^2x + 1$	Teacher illustrated	Teacher illustrated	Teacher illustrated	Teacher illustrated	Teacher illustrated
$sinxtanxcosx$	T: So, after the freak-out moment, what comes next? S: I would probably guess if it was a question. The teacher deliberately asked a student who was struggling to come to the whiteboard. In this case the teacher's response is not connected to any student mathematical thinking. Thus, it falls under what Teuscher et al. (2017) refer to as 'no connection' responding. Warshauer (2014) refers to this type of response by the teacher as *unfocussed* or *vague*.	C: Use the second formula and set it equal to $sinx$. S: Wait, do what now? The struggling student admits that the trigonometric identity $tanx = \frac{sinx}{cosx}$ would be the most appropriate to use; however, the struggling student does not know how to use it.	C: Just write it somewhere, just set it equal to $sinx$. S: What am I writing out? C: The second formula. C: $tanx = \frac{sinx}{cosx}$, so $tanxcosx = \frac{sinx}{cosx}cosx = sinx$ …that did not explain [it] very well. S: No In this case, the uncertainty comes from the struggling student trying to make sense of how the class wants to assist. At times the struggling student seems confused at the directions coming from the class. The didactical praxeology that is	No struggles observed.	No struggles observed.

			informed by the ATD indicates that the teacher removed the cognitive demand on the struggling student by asking a classmate to go to the whiteboard and simplify the expression $sinxcosxtanx$.		
$\frac{sinx + cosx}{sinxcosx}$	T: Are we stuck? C: …Yeah T: Good The teacher noticed after a prolonged period that the students were struggling to get started with simplifying the expression $\frac{sinx+cosx}{sinxcosx}$. Although the teacher proclaimed: "Good", in this instance the teacher does not show any bias towards the students' mathematical ability, but rather uses this opportunity to convey the normality of struggling to the students.	S: Ok, the first thing we did was multiply this by… or *I* did… was to multiply this by $\frac{sinx + cosx}{sinxcosx} sinxcosx$ The struggling student at the whiteboard started to simplify $\frac{sinx+cosx}{sinxcosx}$ erroneously, by multiplying $\frac{sinx+cosx}{sinxcosx}$ with *sinxcosx*, until the class intervened.	S: Is this what you say… $sin^2x + sinxcos^2x$? Although the struggling student received help from the class, she still seemed uncertain about certain mathematical processes.	C: Can we have other numbers that are not squared? The struggling student thought that $sin^2x + cos^2x = 1$ was equivalent. To $sinx + cosx = 1$	No struggles observed.

	Also, the teacher gathered information about the how the students are dealing with the question. This type of questioning, e.g. “Are we stuck”, is called *collecting information.* This type of response by the teacher is referred to as *unfocussed* or *vague* by Warshauer (2014).				

The simplification of the expressions $sin^2x + cos^2x + 1$ and $sinx \frac{1}{tanx}$ resulted in no productive struggles. The reason they are labelled 'teacher illustrated' in Table 4.2 is that at the beginning of the study, the teacher wrote the following trigonometric identities on the top right-hand corner of the whiteboard; $sin^2x + cos^2x = 1$, and $tanx = \frac{sinx}{cosx}$. The teacher then guided the students in simplifying the expression $sinx \frac{1}{tanx}$ by asking them what the most appropriate trigonometric identity substitution would be to use. This type of response by the teacher, in which he guided the students based on their thinking, is referred to as *probing guidance* by Warshauer (2014). After that the students continued independently, easily solving the second question, which was the simplification of the expression $sin^2x + cos^2x + 1$.

Although the struggling student chose the correct identity to simplify *sinxtanxcosx*, the struggling student still stated: "Nah, I really don't know what to do." The teacher still did not lower the cognitive demand on the student by *telling* her the answer at this stage; instead, the teacher asked the class to assist the struggling student. This type of request by the teacher to the class to assist the struggling student is called a *generating discussion* question. After an interaction between the student and the class could not resolve her struggle, the teacher intervened, lowering the cognitive demand on the struggling student by asking one of her classmates to go to the board and write out what should have been done to simplify *sinxtanxcosx*. In this case, therefore, the teacher did not engage in questioning that would demand higher-order thinking, which Menezes et al. (2014) found increased student learning. The students struggled to 'get started' when they were asked to simplify $\frac{sinx+cosx}{sinxcosx}$.

The reader is reminded (from Chapter Two) that Teuscher et al. (2017) expanded Jacobs' (2010) definition of *noticing.* Teuscher et al. (2017) divided the *attending* part of *noticing* into a *general observation* and *student mathematical thinking.* The teacher made a general observation when he asked the class: "Are we stuck?" The struggling student wrote an incorrect operation on the whiteboard, and her body language indicated that she was confused. Stevens (2012) contended that certain body postures in students (for example, hands in the air) indicate confusion. Mason and Singh (2010) proposed a productive conversation – which is what happened in this case, between the struggling student and the class – to eradicate any confusion that the student might experience.

An analysis of the data from the Activity 2 questions at DCAT 1 level is presented in the next section.

4.2.1.4 Distribution of the students' productive struggles in DCAT 1 Activity 2 questions

In this section an analysis of the students' productive struggles during DCAT 1 Activity 2 questions is presented. Activity 2 questions are based on more difficult DCAT 1-level questions and can thus be compared with Activity 1 questions. Table 4.3 below lists examples of students' productive struggles and their frequency during DCAT 1 Activity 2 questions.

Table 4.3: Distribution of students' productive struggles from DCAT 1 Activity 2 questions

Questions	**Types of productive struggle**					
				Misconceptions and errors		
Simplify each of these expressions	**Getting started**	**Carrying out a process**	**Uncertainty in explaining and sense making**	**Expressing misconceptions**	**Expressing errors**	**Total**
$(sinx + cosx)^2 - 2sinxcosx$	0	1	0	0	1	**2**
$tan^2x + 1$	1*	5	0	2	0	**8**
Totals	**1**	**6**	**0**	**2**	**1**	**10**

1 indicates that getting started did not manifest at the start of the question.*

The simplification of the trigonometric expressions $\frac{sinx}{tanx}$, tan^2xcos^2x and $\frac{cosx}{1-sin^2x}$ yielded no productive struggles and was thus excluded from the table.

Usually, the productive struggle 'getting started' occurs at the beginning when the students attempt to solve a question. However, 1* indicates that the students reached an impasse in carrying out a process. The productive struggle 'carrying out a process' occurred the most during Activity 2 for DCAT 1-level questions. The students experienced the most productive struggles when they simplified $tan^2x + 1$. For this simplification, the productive struggle 'carrying out a process' occurred most often.

The simplification of $(sinx + cosx)^2 - 2sinxcosx$ yielded the second-greatest number of examples of students' productive struggles. For the simplification of $(sinx + cosx)^2 - 2sinxcosx$, 'carrying out a process' and 'errors' productive struggles occurred an equal number of times. A struggling student could not expand the binomial $(sinx + cosx)^2$. Bosson-Amedenu (2017) stated that the binomial theorem (and by extension, a squared binomial) is a concept with which students in senior high school experience difficulty. Another struggling student wrote:

$(sinx + cosx)^2 - 2sinxcosx = sin^2x + cos^2x - 2sinxcosx = 1 - 2sinxcosx.$

This solution suggests the student still had difficulties simplifying algebraic expressions of the form $(a + b)^2$; which reaffirms Khuzwayo's (2019) contention that lacking algebraic skills will hamper students when they need to simplify questions relating to trigonometric identities.

In the next section, this study analyses the question that exhibited the most examples of students' productive struggles.

4.2.1.5 An analysis of the question with the highest number of productive struggles from DCAT 1 Activity 2

Most examples of students' productive struggles in the second Activity questions at DCAT 1 level manifested when the students simplified $tan^2x + 1$.

Getting started

Although the class started to simplify $tan^2x + 1$, they reached an impasse when they got to the following stage:

S: *[writes* $\frac{sin^2x+cos^2xsin^2x+cos^4x}{cos^2x}$ *on the whiteboard]*

T: Where do we go from there?

S: *[silence]…*cancel?

T: Where do we go from there?!

T: Is what we have at the bottom *[referring to the students' solution]* simpler or more complex than what we started with?

S: Way more complex!

T: Okay. So we either have to do something [else] *[to simplify the question]* or we have done something wrong [with the current solution].

In this excerpt, the 'getting started' happened when the students reached an impasse after they started to simplify $tan^2x + 1$. However, this study has shown that students might reach an impasse more than once when simplifying trigonometric expressions. In this case, the teacher noticed that the students had reached an impasse. The teacher's response is a question, which leads the class to a dichotomy: is their solution easier or more difficult than the original expression? This guidance from the teacher gave the class introspection into their solution to simplifying the expression. This type of questioning from the teacher, where he probes the students to give an explanation for how to continue their solution, is called a *probing* question. Warshauer (2014) also refers to this type of response from the teacher as *probing guidance.* In terms of the ATD, this question from the teacher informs the students what the most appropriate technique is to use in simplifying the expression. Also, the students lacked *procedural fluency,* which Kilpatrick et al. (2001) described as the effective and malleable, accurate and skilful use of procedures in the simplification of (for example) $\frac{sin^2x+cos^2xsin^2x+cos^4x}{cos^2x}$.

<u>Carrying out a process</u>

A struggling student wrote $tan^2x + 1 = \frac{sin^2x}{cos^2x} + 1 = \frac{sin^2x}{cos^2x} + sin^2x + cos^2x.$
This solution indicates the student's over-reliance on the trigonometric identity $sin^2 + cos^2x = 1$ and his or her inability to have what Kilpatrick et al. (2001) refer to as *procedural fluency;* that is, the ability to apply procedures in appropriate situations.

<u>Expressing misconceptions</u>

T: All right, what are ya'll thoughts?

S: Can you cancel that? Yeah, can you cancel it? So, I know it's not right, but is that it? *[the students used visual cues, wanting to know if the sin^2x's could cancel]*

T: No.

In the excerpt above, the teacher's response is short and clear. This type of response from the teacher is labelled *unfocused* or *vague* by Warshauer (2014). That is, the teacher did not direct the student to a certain strategy; neither does it build on student thinking. The teacher provides a suggestion that is too broad to be helpful. Again, the teacher used a *generating discussion* question to elicit help from the class. Also, in this excerpt, the student lacked *conceptual understanding* of when to cancel out terms and the *procedural fluency* to do so. Later, during the struggle to simplify $tan^2x + 1$, some students still seemed confused about the process of cancellation. When the students reached a second impasse, the teacher did not remove the cognitive demand by giving a direct answer. Instead, he responded by saying "That's a great question", thus leaving the cognitive demand on the class. The following conversation ensued:

S: Okay, $sin^2 + cos^2x$ = 1. But where do you go from that?

T: That's a great question.

S: So, we can cancel out?

T: No, they cannot cancel; they can only cancel when we're multiplying.

The cancelling error was made more than once. There was evidence that even after the teacher verbally explained under what conditions cancelling can take place, some students still did not fully understand the concept. Immediately after the teacher explained under what conditions one may cancel, the following conversation took place.

S: *[writing on whiteboard]* $\frac{sin^2x}{cos^2x} + \frac{cos^2xsin^2x}{cos^2x} + \frac{cos^2xcos^2x}{cos^2x}$

T: Okay, what do we get when we do that?

S: *[writing on whiteboard]* $\frac{sin^2x + cos^2xsin^2x + cos^4x}{cos^2x}$

T: Where do we go from there?

S: Cancel? *[followed by silence]*

T: Where do we go from there?

S: I say, cancel out the cos^2x in front of the sin^2x.

T: No! They cannot cancel. They can only cancel when we are multiplying.

In this case, the proximity of the cos^2x's in the denominator and numerator suggests why the student thought the cos^2x's could cancel each other out. However, this attempt at cancellation further suggests that operations on fractions in trigonometry are still an area of great confusion to the students. The implication is that the students did not fully understand the concept of fractions in earlier grades, and thus their lack of adaptive reasoning is a worrying aspect. Khuzwayo (2019) investigated student understanding of trigonometric identities and observed that high-school students lack basic algebraic skills such as manipulating fractions. Similarly, Orhun (2010) reported that first-year calculus students performed poorly on trigonometric expressions. Orhun (2010) found that students' lack of *adaptability* of old knowledge to new knowledge stifled their progress in trigonometry. For example, students might know that $sin30^0 = \frac{1}{2}$ but might be confused by *sin30;* that is, the *sine* of 30 *radians.*

In this regard, to elicit justification or explanation for the students' work, the teacher used *probing* questions. Moreover, intervention by the teacher – by explaining to the students under what conditions they could cancel – did not resolve their struggles. Warshauer (2014) refers to this type of response from the teacher as *probing guidance.*

Next, this study summarizes the rest of the students' productive struggles from the DCAT 1 Activity 2 questions. The simplification of $tan^2x + 1$ has already been discussed in detail and is thus excluded.

4.2.1.6 Illustrations of the examples of students' productive struggles in DCAT 1 Activity 2 questions

Table 4.4 below gives a summary of the recorded examples of students' productive struggles during DCAT 1 Activity 2 questions, and the evidence that supports their classification as productive struggles, as well as how the teacher and class responded to these productive struggles. Since Table 4.3 contains only two questions from the DCAT 1 Activity 2 questions, and the simplification of $tan^2x + 1$ has already been discussed in detail, only one question was analysed and is presented in Table 4.4.

Table 4.4: Summary of examples of students' productive struggles from DCAT 1 Activity 2 questions

Questions	**Types of productive struggle**	
		Express misconceptions and errors
Simplify each of these expressions	**Carrying out a process**	**Misconceptions**
$(sinx + cosx)^2 - 2sinxcosx$	S: ($sin^2x + cos^2x$)-*2sinxcosx = 1*-2*sinxcosx*. That's what I got. T: Ok C: Can you divide by 2? T: You can divide by 2, but what does that help you? The teacher response in this instance elicits further thinking by the student. Although the teacher initially gave a direct response: "No", this statement is immediately followed by a question from the teacher. In terms of the ATD, the teacher requires the student to produce the *technology* that would support the *technique* employed by the student.	S: *[comes forward to the whiteboard and writes]:* $sin^2x + sinxcosx + cosxsinx + cos^2x$*= 1+2sinxcosx -2sinxcosx=1* S: Is that right? T: What is his final answer? S: One. T: I don't know…is it? S: [*Throws her hands in the air in disgust]* The questioning by the teacher again places the onus on the class to help solve the question. The class resolved this misconception by writing out all the steps that are necessary to expand the binomial that was squared. By asking the student: "…is it?" the teacher indirectly wants the student to justify their answer. This type of questioning is called *broadening thinking.* Warshauer (2014) refers to this type of response from the teacher, where he encourages student self-reflection, as *probing guidance.*

The productive struggles 'getting started', 'uncertainty in explaining and sense making' and 'errors' were not observed during DCAT 1 Activity 2 questions and were consequently removed from Table 4.4.

Tainio and Laine (2015) suggest that teachers should communicate that incorrect answers make an important contribution in mathematics. Table 4.4 shows the student incorrectly expanded $(sinx + cosx)^2 - 2sinxcosx$. The teacher did not reprimand the student or criticise the answer, merely responding with "Okay". During the struggle that followed, again the teacher did not intervene. He merely asked: "Is it?" and thus indirectly posed the question to the class. Tainio and Laine (2015) refer to this idea of directing an incorrect response to a different student as 'turn-allocation'.

In the following section, this study compares the productive struggles experienced by the students in DCAT 1, Activity 1 and Activity 2. The comparison relates to the types and frequencies of students' productive struggles, as well as the algebraic errors they made.

4.2.1.7 Similarities and differences of students' productive struggles between DCAT 1 Activity 1 and Activity 2 questions

Between the first and second activities there was a significant decrease in the number of students who had issues with 'getting started'. The 'getting started' issue in Activity 2 questions arose in conjunction with the notion of what is considered 'simplification'. The students 'simplified' an expression, only to realise that their answer was 'worse' than what they had started with. The teacher noticed this, and mentioned it. One of the students complained: "How can the struggle be productive if you don't know what you're doing?" Thus, the students' 'getting started' issue shifted from the beginning of simplifying $\frac{sinx+cosx}{sinxcosx}$ to the end of simplifying it.

Between Activity 1 and Activity 2 questions, the number of examples of students' productive struggles associated with 'carrying out a process' remained nearly the same. The students struggled with various algebraic processes. These processes included manipulating equations; for example, the students struggled to change $sin^2x + cos^2x = 1$ to $cos^2x = 1 - sin^2x$. Then they experienced difficulties with division of fractions. Lastly, the students struggled with division questions where the numerator was a binomial and the denominator a monomial, for example $\frac{1-sin^2x}{cos^2x}$. During Activity 2 questions the students struggled with expanding binomials, cancellation errors, manipulating algebraic fractions and factorisation errors. Thus, it seems that some of the struggles from Activity 1 questions had not been resolved by the time the Activity 2 questions were attempted.

The next section discusses the teacher's meeting with the researcher about the students' struggles in the classroom.

4.2.1.8 Highlights from the post-teaching meeting between the teacher and researcher regarding DCAT 1 Activity 1 and Activity 2 questions

This section discusses the reflections from the post-activities meeting between the teacher and the researcher, and their importance to the study. The reader is reminded that the role of the teacher was to facilitate the activity questions and support the students when they struggled to simplify trigonometric expressions and prove trigonometric identities. The teacher's role also included pre- and post-activity meeting obligations. At these pre- and post-activity meetings the teacher gave the researcher valuable information regarding classroom dynamics that was not captured in the video recordings.

The researcher and the teacher met before school commenced in the morning. All meetings were transcribed verbatim. R = researcher and T = teacher. The following themes emanated from this post-activity meeting:

Over-reliance on replacing 1 with $sin^2x + cos^2x$

At this point of the study, the students' inability to perform basic algebraic manipulations appeared to be stifling their progress in trigonometry. For example, in one question from DCAT 1 Activity 2 (shown in Figure 4.1 below), the unnecessary substitution of '1' with $sin^2x + cos^2x$ made the simplification of $tan^2x + 1$ 'worse'.

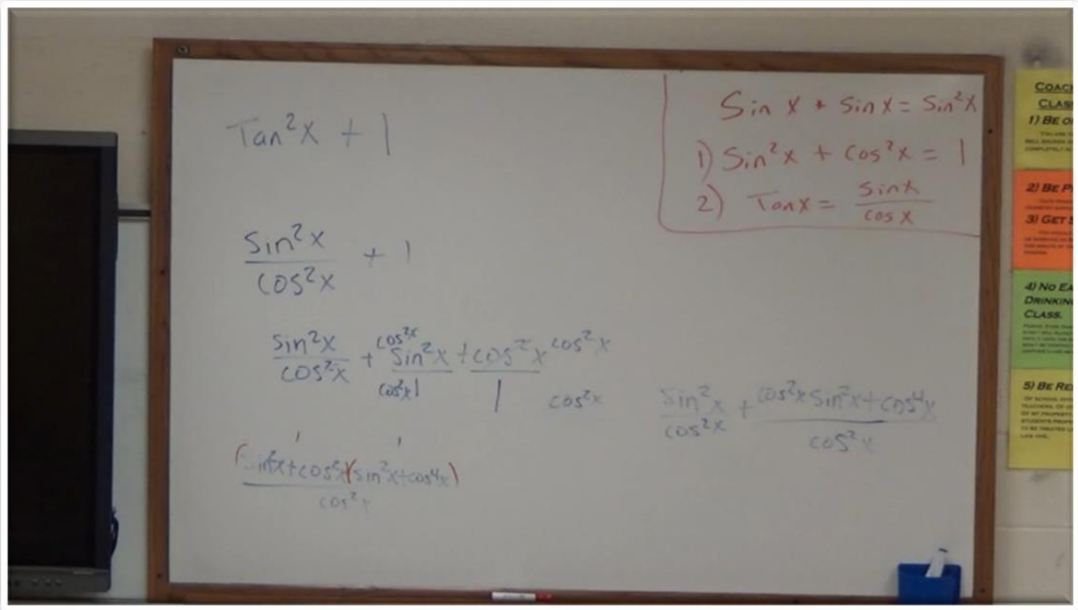

Figure 4.1: Student work during DCAT 1 Activity 2 questions

Moreover, in the last step in Figure 4.1 above, one of the students erroneously grouped sin^2x and cos^2x, as well as sin^2x and cos^4x.
The following excerpt provides evidence of the students' over-reliance on $sin^2x + cos^2x$:

R: *[Laughing]* Talking about the 1… so where do you think the reversal of 1 being $sin^2x + cos^2x$, to 1 being something over itself, comes from? Because last week, every single 1 was $sin^2x + cos^2x$.

T: But, so they, um… you'll see in the video, when we send a student to the board *[to simplify $tan^2x + 1$]*, she did, like, four different things; and three of them were changing 1 to $sin^2x + cos^2x$, but one of *[the attempts]*, she just left the 1. Um… *[referring to the students' overreliance on $sin^2x + cos^2x$ as a replacement for 1]*

R: I mean, so, why do you think… where does it come from? Do you think it's just because that's the thing *[the identity]* they started off with?

T: Um…

R: I mean, the first identity *[$sin^2x + cos^2x = 1$]*. So, everything. They assume they must *[change to $sin^2x + cos^2x = 1$]*?

T: Yeah. But also, like, they don't know… they haven't discovered what they can do with that. *[The teacher means the students did not realise that you can manipulate a trigonometric identity in the same way you manipulate an algebraic equation]*

R: Okay.

T: Because, as we've talked about yesterday… without me prompting them to say, well…
It could be $sin^2x = 1 - cos^2x$, or cos^2x = 1 – sin^2x… we never talked about how that can be manipulated.

This last statement from the teacher means that students did not yet see the relationship between $sin^2x + cos^2x = 1$ and $sin^2x = 1 - cos^2x$ or $cos^2x = 1 - sin^2x$; although the students might be able to manipulate $x^2 + y^2 = 1$ to get $x^2 = 1 - y^2$. Thus, the teacher's reflections agree with Price and Van Jaarsveld (2018), who contended that trigonometry is often misunderstood because of the strong connections between algebra, geometry, and graphical ways of reasoning. The initial phase of this study indicated the importance of the teacher *attending* to the student struggle, *interpreting* the struggle, and appropriately *responding* to the struggle. In this regard, the teacher acknowledges that he mainly focused on *directed* and *probing guidance.*

The advantages of working in a group

In their investigation into teaching higher-level mathematics, Yadgarovna and Husenovich (2020) found that small groups lead to better mastery of content.

In one of the findings from the comparison between the students' productive struggles during Activity 1 and Activity 2 questions, the researcher noted that there was a significant increase in speed in 'getting started' between the two activity questions – the students took less time to get started with an activity question. During the post-teaching meeting, the teacher remarked: "…but I think they are starting to get there". Furthermore, the assertion from the teacher that the students did not know that they could change the subject of the formula agreed with the observation made in this study when comparing student responses in Activity 1 and Activity 2 questions.

What was encouraging was that the teacher noticed that the students were starting to collaborate. The following post-teaching meeting excerpt shows evidence of the students' collaboration.

R: Okay. So, your struggle is just for them, um, to struggle some more, and [to] leave them alone.

T: Right, I guess, like, it was because I fully anticipated – on the $tan^2x + 1$ – having to give them some sort of…

R: Guidance?

T: Guidance.

R: Okay.

T: …but I sat here, and they kept having to… they were debating about it, and they kept having meaningful conversation, and, you know? It wasn't, like, a dead-end conversation. So they eventually – as a group – got it.

R: Okay. That's nice.

Jelatu, Kurniawan, Kurnila, Mandur and Jundu (2019) found that students perform better in trigonometry if they collaborate rather than learning the traditional way. In fact, collaboration between students, together with the right guidance and support from the teacher, is what this study encouraged in Chapters One and Three. However, "perseverance is only useful when students have sufficient prior knowledge and metacognitive abilities to make their struggles potentially productive" (Star, 2015, p.10). Thus, the inability of students to perform basic algebraic manipulations may restrict the advantages of working in groups. Moreover, the teacher guidance took the form of careful questioning and noticing. Furthermore, the teacher adopted a *professional noticing* approach when he said: "Right, I guess, like, it was because I fully anticipated…"

The purpose of NEA was discussed in detail in Chapter Two. The next section discusses how this study used NEA to ascertain which productive struggles remained unresolved in the assessments.

4.2.1.9 Using NEA to analyse students' mathematical thinking in Assessment 1

In this section, the error analysis developed by Newman (1983) was used to establish which productive struggles remained unresolved after Activities 1 and 2. In Chapter Two, this study explained how analysis of student errors can help teachers to comprehend how students learn, and to understand the students' thought processes. For example, Mensah (2017) used the NEA model to investigate senior high school students' errors made when learning trigonometry.
Each question in the assessment was carefully analysed, and coded (using Newman's 1983 Error Hierarchy Model) as: *read, comprehension, transformations, process skill, encoding* or *other*.
If a wrong answer without reasons was given, or if the student set the expression equal to zero, the error was coded as a *comprehension hierarchy error.* If an error was made immediately after the question was rewritten, it was coded as a *transformation hierarchy error*. If the error was not made at the *transformation hierarchy*, the researcher investigated whether it had been made in the *process skills hierarchy;* and if no errors were identified at any of these stages the researcher investigated whether the final answers were *encoded* correctly. 'Encoding' errors included 'not simplified' or if the answer had become more complicated than the question, or algebraic errors such as $sinx + sinx = sin^2x$. The term '*other'* covered errors that could not be described as *comprehension* or *reading, transformation*, *process skills* or *encoding error.* In the following section, the distribution of NEA is presented.

4.2.1.10 Categorisation of the errors using NEA

Table 4.5 presents all the errors made by the students during the first assessment on DCAT 1 activity questions. The number in each cell indicates the number of times each type of error was made for each question.

Table 4.5: Distribution of NEA types among assessment 1 questions

Assessment question	**Type of NEA**					
	Comprehension	Transformation	Process skills	Encoding	Other	Total
$\frac{cosx}{\frac{1}{tanx}}$	0	1	4	2	0	**7**
$\frac{sinx}{1-cos^2x}$	0	1	0	3	0	**4**
$\frac{sin^2x+cos^2x}{2sinx}$	0	1	0	0	0	**1**
$\frac{cos^2x}{1-cos^2x}$	1	0	0	2	0	**3**
$\frac{1}{cos^2x}-1$	2	0	14	3	0	**19**
$\frac{1}{tan^2x}sin^2x$	0	3	5	4	0	**12**
$tanxcosx + sinx$	2	0	0	3	0	**5**
Total	**5**	**6**	**23**	**17**	**0**	**51**

During Assessment 1 the students made the most errors at the *process skill* hierarchy, accounting for 23 out of 51 errors. Most errors at the *process skill* hierarchy were made when the students simplified the expression $\frac{1}{cos^2x}-1$. The errors at the *process skill hierarchy* included cancellation errors, manipulating algebraic fractions, and the unnecessary use of $sin^2x + cos^2x = 1$.

The second-greatest number of errors was at the *encoding* hierarchy, accounting for 17 out of 51 errors. Most errors at the *encoding hierarchy* were made when the students simplified the expression $\frac{1}{tan^2x}sin^2x$. This would suggest that some students still struggled to represent their work mathematically.

Notably, there were no *reading errors* made on the assessment, and consequently this NEA hierarchy was omitted from the table. Also, the students made the least number of errors when they simplified the expression $\frac{sin^2x+cos^2x}{2sinx}$. In contrast to this study's observation, Wahyuni and Widayanti's (2020) investigation into errors made by students when the students were asked to solve story problems in trigonometry found that most of the NEA occurred at the *encoding hierarchy*, followed by errors at the *process skill hierarchy*.

4.2.1.11 Findings from the focus group interview on DCAT 1 assessment

The main purpose of the focus group interview was to obtain evidence on whether students' productive struggles that were observed during DCAT 1 learning activities were resolved during the first assessment. The focus group interview thus gave the researcher a deep understanding of the nature of the errors committed by the students.

The focus group met during the lunch break to discuss Assessment 1. Five members were present: Brigitte, Aud, Maud, Amy, and Pam. Each member received their Assessment 1 scripts back to reflect on their answer choices. The reader is reminded that the focus group represented the participants academically. Thus low-, medium- and high-ability students were approached (and agreed) to participate in the focus group. Two low- and medium-ability and one high-ability students were approached by the researcher to see if they were willing to participate in the focus group. All the students who were approached to participate in the focus group agreed to it. The focus group knew one another very well and felt comfortable in raising their views when prompted by the researcher or other members of the focus group. Assessment 1 can be found in Appendix 6. Themes emanating from the focus group interviews were analysed. In the following interview excerpt, R = researcher.

Cancellation errors

Three of the five focus group members still struggled with algebraic concepts, with cancellation errors. The students did not know under what conditions cancellation may take place. For example, in Figure 4.2 below, Amy's final answer to Question 1 was correct, however, her procedure for calculating the answer was not correct. In her first step, Amy cancelled the $cosx$ in the numerator with one of the *cosx*-es in $sin^2x + cos^2x$.

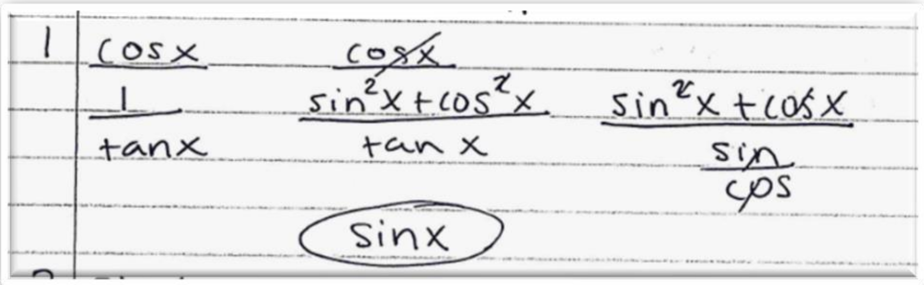

Figure 4.2: Amy’s response to Question1 on the assessment

Amy’s cancellation error in her simplification $\frac{cosx}{\frac{1}{tanx}}$ is rooted in her inability to manipulate arithmetic fractions. In her study of students’ understanding of trigonometric identities, Khuzwayo (2019) observed similar errors. She noted that six out of eight students working with fractions could not apply the relevant algebraic operations. Recall from Chapter Two that Chevallard (1992) introduced the ATD, which (by means of *praxeologies*) reveals how knowledge is acquired. The *practical block* (‘know how’) consists of the *type of task* and *technique*, whereas the *theoretical block* (‘know why’) comprises the *technology* and *theory.* In Amy’s case, her implementation of the *technique* to *simplify trigonometric expressions* was not accurate. However, the *technology* – that is, the justification for the *technique* – was correct. That means the idea of using cancellation was correct, but the correct substitution of $tanx$ with $\frac{sinx}{cosx}$ must still be made.

When the simplification of the expressions $\frac{cosx}{tanx}sinx$ and $tan^2x + 1$ was analysed, some students struggled with cancellation errors in these examples. The following interview excerpt provides evidence of Amy’s misconception regarding cancellation.

Pam:	I did number 1 *[question 1]* wrong.
Brigitte:	I did number 1 and 2 *[question 1 and 2]* wrong, because I marked them out and went back. *[Brigitte is arguing that the reason she answered question 1 and question 2 incorrectly is because she erased question 1 and 2, then continued with question 3 but went back to reattempt questions 1 and 2].*

Pam: I'm looking at my answers and trying to figure out how in the world I got it *[the answers to question 1 and 2]*.

Brigitte: I don't think I could have done that *[incorrect algebraic manipulation]* but I did on number 1 *[question 1]*.

Amy: Oh, I did the same thing. *[addressing the whole focus group]* What did you get?

Brigitte: I got $sinx$ *[as the solution to question 1]*.

Amy: Me too. But I cancelled out something.

R: Amy, you look perplexed – what do you say?

Amy: I should have got $\frac{1}{sinx}$. Is that what you all got for number 1 *[question 1]*?
Like, did you do not get $sinx$?
I should have got $\frac{1}{sinx}$; I don't know why I got $sinx$. Because I marked out *[referring to cancellation]* the cosines and I should have put a one on top, but I didn't. I feel dumb.

Pam, Brigitte and Amy's misconceptions about mathematical concepts are illustrated in the excerpt above. However, none of them realised their deep-seated misconceptions when they gave vague reasons for their mistakes. Luneta (2013) contended that such misconceptions are mathematical ideas that the students should have learned in earlier grades. The misconceptions and errors made by Pam, Brigitte, and Amy are a common theme in Chapter Four.

The following interview excerpt presents evidence that other members of the focus group also struggled with cancellation errors. At this stage, the focus group members were exchanging thoughts on why they had solved questions 1and 2 in particular ways. The focus group also shared their frustration about why they had chosen certain techniques to solve question 1 and 2.

Maud: I was cancelling when I'm not supposed to.

R: What did you say, Maud? You were cancelling when you are not supposed to?

Maud: ...so I could just get done with it.

R: So, you didn't focus on it, you just did it because you had to get done with it?

Maud: I thought that I could cancel right there *[pointing at her solution]*, but now I know I can't.

Brigitte: Yeah, I cancelled at places that I shouldn't have cancelled.

Amy: What number *[question]* are you talking about?

Amy: Number 1 *[question 1]*. But I got those cosines.

Brigitte: I did too. I wasn't supposed to.

Brigitte: This was ... I think this was all before, when we knew what we were doing, but we practised before and were ready for this. I guess we're not 100% comfortable with everything yet.

Pam: I am sure I was making up stuff that I wasn't sure *[about]*.

Brigitte reaffirmed what the teacher had said during his post-teaching meeting. As a reminder: the teacher contended that at the beginning of the study, the students did not yet know that they could apply algebraic manipulation when simplifying trigonometric expressions. He then said: "Yeah. But also, like, they don't know… they haven't discovered what they can do with that."

Over-reliance on $\boldsymbol{sin^2x + cos^2x = 1}$

The focus group interview reveals that the overuse of $sin^2x + cos^2x = 1$ stems from two issues. First, when the study started, the trigonometric identity $sin^2x + cos^2x = 1$ was the first identity that the students used. Second, the students did not fully grasp the concept of 'simplification'.

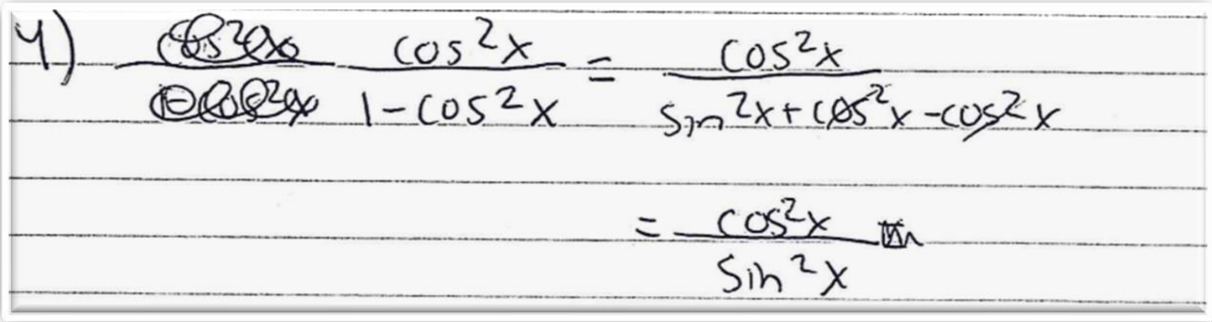

Figure 4.3: Brigitte's response to Question 4 on the assessment

It is standard practice to replace $1 - cos^2x$ with sin^2x. However, in Figure 4.3 above we see that Brigitte unnecessarily replaces 1 with $sin^2x + cos^2x$ when simplifying $\frac{cos^2x}{1-cos^2x}$. The except below from the subsequent interview provides evidence of the overuse of the trigonometric identity $sin^2x + cos^2x = 1$.

R: So in essence, if you look back at the first assessment, what do you think… okay, *where* do you think you have been very naïve?

Pam: That you always have to change the 1. Because it's not true – you don't.

R: So why did you always change the 1 back then?

Amy: I thought it would be some complex answer that you should get – that's the way to go.

Brigitte: That is the rule that we use the most.

R: $sin^2x + cos^2x = 1$?

Brigitte: Yeah.

R: So, you thought that everything 1 had to be *[replaced with]* sin^2x+cos^2x=1?

Brigitte: Yes. The $\frac{cos^2x}{cos^2x}$ in number 5 *[question 5]* I think, where you change to 1 on the outside? *[referring to the -1 in* $\frac{1}{cos^2x} - 1$*]*.

R: Yeah?

Brigitte: I didn't feel like you could do that until later.

The focus group members were all in agreement with Brigitte's last statement.

According to Kilpatrick et al. (2001), conceptual understanding is the comprehension of mathematical ideas. In response to Amy's question as to whether Brigitte had written $sinx$ as her answer, Brigitte replied: "That is pretty much what I got – I got $\frac{1}{sinx}$". Brigitte's answer indicates that for her, the distinction between $\frac{1}{sinx}$ and $sinx$ was not well developed.

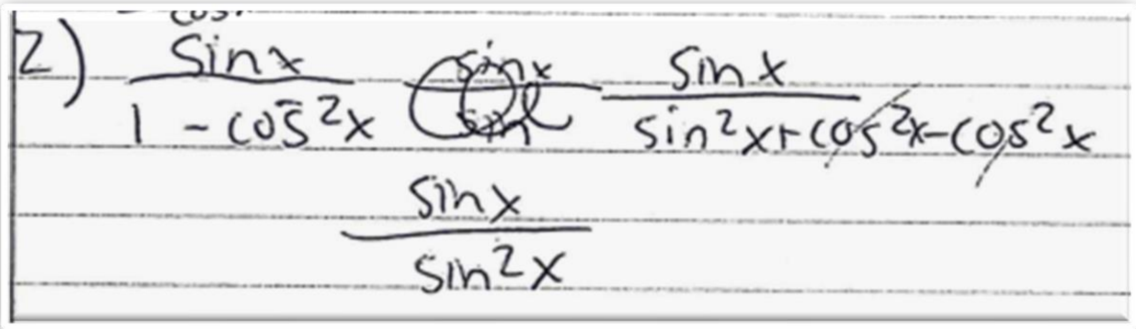

Figure 4.4: Brigitte's response to Question 2 of the assessment

Moreover, Kilpatrick et.al. (2001) defined adaptive reasoning as the capacity for rational thought, reflection, clarification, and validation. In Figure 4.4 above, Brigitte struggled to see, for example, that $\frac{x}{x^2}$ and $\frac{sinx}{sin^2x}$ are simplified similarly. Additionally, she could not see the similarity between $\frac{1}{\frac{a}{b}} = \frac{b}{a}$ and $\frac{1}{\frac{sin^2x}{cos^2x}} = \frac{cos^2x}{sin^2x}$. The finding from the focus group interview is thus in agreement with Nanmumpuni and Retnawati (2021), who in their investigation into students' difficulties with conceptual trigonometry questions found that students found it difficult to apply mathematical concepts such as those for fractions when solving trigonometry questions.

The following interview excerpt is evidence of Brigitte's problem.

Pam: What did you get for number 2 *[question 2]*?

Brigitte: $\frac{sinx}{sin^2x}$

Pam: I got… *[Pam was distracted and did not complete her sentence].*

Maud: I got $\frac{1}{sinx}$.

Amy: I got $\frac{1}{sinx}$ as well.

Pam: But what I'm thinking now is that $1 - cos^2x$ should have been sin^2x and we expanded *[simplified]*.

Amy: Because I put sin^2x *[in the denominator as well]*.

Brigitte: That makes sense. I didn't expand it out *[simplify it]*.

R: What was problematic about number 4 *[question 4]*?

Amy: I got negative one *[as the numerator]*.

Pam: I got $\frac{-1}{sin^2x}$.

Maud: I think I cancelled at places that I shouldn't have cancelled.

Pam: I didn't have any work *[to show]*.

Brigitte: I didn't put the $\frac{1}{\frac{sin^2x}{cos^2x}}$ to get *tan*^{2}*x*, I just left it as cos^2x over sin^2x because I didn't think you could.

The focus group interview shed light on why errors to do with the *process skills hierarchy* and the *encoding hierarchy* dominated Table 4.5. Some members of the focus group continued to struggle with the same mathematical operations as when they completed Activities 1 and 2 under DCAT 1. These mathematical operations are the manipulation of fractions, simplification errors and cancellation errors. In their investigation into student difficulties in solving trigonometric equations and identities, Rohimah and Prabawanto (2019) found that students lacked the skills to complete basic algebraic calculations and computations. Hidayati (2020) also found that students solving trigonometric questions lack knowledge of basic mathematical operations. There were no reading errors observed during Assessment 1, so the students were clear about the objective of the question at hand. However, the findings from the focus group interview reveal some deep-seated misconceptions about the manipulation of algebraic fractions. In the incorrectly answered questions, from an ATD perspective, the *technique* used by the students to solve the *task type* was not incorrect per se; however, the implementation of the *technique* was inappropriate. For example, for Brigitte's response to question 2 on the assessment, she correctly replaces 1 with $sin^2x + cos^2x$, but fails to simplify $\frac{sinx}{sin^2x}$.

Even though the concept of simplification of trigonometric expressions was new to the students – and this study does agree with McDonald (2010), who stated that "…misconceptions occur when a new idea has no connection with existing knowledge and therefore becomes impossible to understand" (p.34) – the teacher had explained to the class under what conditions cancellation of terms may occur. The ATD theory, originated by Chevallard (1992), posits that the *technology* component of the *theoretical block* justifies the *technique* used by the students to solve the questions in the assessment. The *knowledge block* under ATD is held primarily by the teachers. Thus, the significance of the findings from the focus group interview concurs with Baidoo (2019) that teachers should take note of these misconceptions and adjust their instruction strategies to minimise them.

4.2.2 Data Analyses on DCAT 2 – Application-level activities

This section presents an analysis of the activity questions at the DCAT 2 level. Similarly to DCAT 1, two activities (Activity 1 and Activity 2) were analysed at the DCAT 2 level.

4.2.2.1 Illustrations of the students' productive struggles with DCAT 2 Activity 1 questions

Table 4.6 below summarises examples of students' productive struggles when the students simplified the trigonometric expressions at the DCAT 2 level in the left-most column of the table. The analysis was completed by observing and analysing the video recordings of the questions attempted by the students in the classroom. The data was transcribed by watching short clips and making sure all data in the video was transcribed verbatim. This transcription process was iterative, to ensure no data from the video recording was missed out.

The number in each cell indicates the number of times a specific type of productive struggle was observed on the video recordings. The total on the right side of the table tallies the number of examples of students' productive struggles that were observed *per question*. The totals at the bottom of the table tally the total number of productive struggles for DCAT 2 activity 1 questions.

For the rest of the questions this study used the Atlas.ti software to determine how many times a type of productive struggle was observed when the students simplified the expressions.

Since the questions in assessment 2 were at the DCAT 2 level, the analysis gives an indication of the progress made by the students to resolve some of the productive struggles that were present during DCAT 1 activities. Consequently, this study could compare if those initial productive struggles were resolved with more difficult questions.

Table 4.6: Distribution of students' productive struggles from DCAT 2 Activity 1 questions

Questions	**Types of productive struggles**				
			Express misconceptions and errors		
Simplify each of the following expressions	**Getting started**	**Carrying out a process**	**Expressing misconceptions**	**Errors**	**Total**
$sinxcos^2x - sinx$	1	1	7	1	**9**
$\frac{cos^2x}{1 - sinx}$	0	3	4	0	**7**
$sinx + \frac{cosx}{tanx}$	0	1	0	0	**1**
Total	**1**	**5**	**11**	**1**	**17**

Table 4.6 does not contain the 'Uncertainty in explaining and sense making' column, since this productive struggle was not observed during DCAT 2 Activity 1 questions. Likewise, the trigonometric expressions $\frac{sinx}{cos^3x} \div \frac{tanx}{cos^3x}$ and $\frac{2cosx}{sinx} + \frac{2sinx}{cosx}$ were excluded from this table, since no productive struggles were observed from it.

The simplification of the expression $sinxcos^2x - sinx$ produced the most examples of students' productive struggles, predominantly 'expressing misconceptions', which was the productive struggle type that occurred the most. On the other hand, the trigonometric expression $sinx + \frac{cosx}{tanx}$ only produced one productive struggle.

4.2.2.2 An analysis of the question with the highest number of productive struggles from DCAT 2 Activity 1

The simplification of the expression $sinxcos^2x - sinx$ contained the most examples of productive struggles from Activity 1. In the context of DCAT 2 activities, S = struggling student, T = teacher and C = classmate(s).

Getting started

S: *[writes]* $sinx \cdot cos^2x - sinx$.

T: Okay; so, what did you do there?

S: Separate them and put [in] the multiplication sign.

T: Okay, what are you going to do now?

S: I don't know.

In this excerpt, the struggling student had difficulty getting started. The teacher did not lower the cognitive demand on the student by giving him the answer. Instead, the teacher put the burden on the struggling student to come up with a solution. That is, the teacher wanted to guide the student, based on student thinking. This process is known as *focusing* and was discussed in Chapter Two. The type of questioning by the teacher is known as *orienting* and *focusing.* This type of teacher response (where the teacher's response is based on student thinking) is labelled *probing guidance* (Warshauer, 2014). Since the student was struggling to get started with the activity question, he or she lacked *conceptual understanding* of the question.

Carrying out a process

C: Give us a little push! Can we divide by $sine$?

T: You cannot divide by $sine$.

C: Can you square them?

S: *[writes]* $sinx \cdot (cos^2x - sinx)$

T: That's not a bad idea… it's not exactly what you do, but it's not a bad idea.

In the excerpt above, the process of factorisation still seemed to be a stumbling block for most of the participating students. This excerpt exhibited two types of responses from the teacher. In the first instance he tells the student: "You cannot divide by *sine*." This type of response where the teacher supplies the student(s) with information is labelled *telling* and the telling might have been a problem for the productive struggles. Secondly, the teacher uses *directed guidance*, re-directing the students to a strategy that is aligned with the teacher's thinking by saying: "…it is not exactly what you do..." Again, the teacher places the onus on the students to solve the question. This excerpt also illustrates the students' lack of *procedural fluency*, since they did not fully understand the concept of factorisation.

Expressing misconceptions

S:	*[writes] –* $sinx \cdot (cos^2x - sinx)$ I was going to do this *[indicating that he wants to distribute the* $sinx$ *into the parenthesis].* $sinxcos^2x - sinxsinx$ *[Inexplicably, the student inserts -1 in front of* $sinx$*:* $-1sinx \cdot (cos^2x - sinx)$*]*
T:	We can't randomly throw parentheses in there!
S:	That's what I was wondering.

Throughout this excerpt, the teacher maintained cognitive demand by not explicitly showing the students how to factorise. On the other hand, randomly inserting -1 in front of $sinx$ to produce $-1sinx \cdot (cos^2x - sinx)$ indicates that the student had an incomplete understanding of factorisation. The teacher resolved the student's struggle by *telling* him: "We can't randomly throw parentheses in there!" Again, the lack of *procedural fluency* prohibits the student from solving the question.

Error

T:	All right… *[writes on the right-hand side of the whiteboard]* $3x^2 + 2x$. If you were told to factor that, how would you factor that?
S:	I [would] probably take the *x* out first. $x(3x + 2)$
T:	So, what is *x*?

S: What do you mean?

C: Oh! You take $sine$ [of] out the whole question… never mind, you can't.

Khuzwayo (2019) argues that, for example, reasoning in which students cannot see the resemblance between $(xy^2 - x)$ and $(sinxcos^2x - sinx)$ is common in students who lack *adaptive reasoning.* Because of this student's poor algebraic skills, the teacher resolved this struggle mainly by answering questions that were posed by the other students who were trying to simplify $sinxcos^2x - sinx$. However, at some point during the activity question the teacher was compelled to give an analogous algebraic question to $sinxcos^2x - sinx$, to point the struggling students in the right direction to simplify $sinxcos^2x - sinx$. In this regard the teacher asked *collecting information* questions (in this case, the teacher wanted to know what the common factor was) as well as *linking* and *applying* questions (linking *x* as the common factor with *sinx* as the common factor). In this regard, the teacher responded by *telling, probing,* and *directed guidance.*

After the teacher realised that the students do not know how to factorise $sinxcos^2x - sinx$, he wrote the following expression on the board: $3x^2 + 2x$. Then he asked the class: "If you were told to factor that, how would you factor that?" The class responds by saying that they would take out x. The student at the board then wrote *x(3x + 2).* In this case, it was imperative for the teacher to re-explain the process of factorisation to the students. Although this action by the teacher might seem to have been counterproductive, Warshauer (2014) stated that direct action by a teacher can lead to a community of understanding and resolve students' struggling without depriving them of the opportunity to learn. However, in this case the teacher's response was to channel the students towards the correct answer by giving them an analogous question.

The teacher then asked the struggling student what *'x'* was. The student made an error by replying: "Oh! You take $sine$ out [of] the whole problem… never mind, you can't." The students exhibit a lack of the *adaptive reasoning* that is required to see the resemblance between $3x^2 + 2x$ and $3sin^2x + 2sinx$.

The next section summarises and discusses questions from DCAT 2 Activity 1 questions other than the simplification of the expression $sinxcos^2x - sinx$.

4.2.2.3 Illustrations of students' productive struggles from DCAT 2 Activity 1 questions

Table 4.7 below is a summary of examples of productive struggles experienced (and the evidence to support their categorisation) during DCAT 2, Activity 1 questions as well as how the teacher and class responded to these struggles. The simplification of $sinxcos^2x - sinx$ exhibited examples of most of the types of productive struggle, and was thus excluded in the summary because it has already been analysed.

Table 4.7: Summary of examples of students' productive struggles from DCAT 2, Activity 1 questions

Questions	Types of productive struggles	
		Expressing misconceptions and errors
Simplify each of the following expressions	**Carrying out a process**	**Misconceptions**
$\frac{sinx}{cos^2x} \div \frac{tanx}{cos^2x}$	Teacher illustrated	Teacher illustrated.
$\frac{cos^2x}{1-sinx}$	S: $\frac{cos^2x}{1-sinx} = \frac{cos^2x}{sin^2x+cos^2x-sinx}$. I then cancelled the $cos^2x's$, I don't know if I can do that? In this excerpt the student is unsure how to simplify an expression, and particularly under what conditions to cancel. The teacher resolved this struggle with *directed guidance by* stating: "Guys, you can cancel if it's a term like a multiply term at the bottom. But if we are adding and subtracting, we can't cancel." *[the teacher is trying to convey to the students 146e146x+s can only occur if terms in the numerator are multiplied with one another and terms in the denominator are multiplied with one another, but not if those terms are added or subtracted from one another.]* In this case, there was no questioning from the teacher.	C: Can we pull a sine out? [*meaning 'factorise $sinx$ as a common factor'*] C: [*Directing their attention to the teacher]* So, can we pull out a *sinx?* T: So… there wasn't a *$sine$ [sinx]* in each term. So you couldn't. In this excerpt, the student is confused about how to determine a common factor when performing factorisation. In this instance the teacher resolved the struggle by telling the students the answer, to eradicate any confusion that the student had about factoring a common factor. However, the cognitive demand on the student was lowered.

$Sinx + \frac{cosx}{tanx}$	S: *[writes]* $sinx + \frac{\frac{cosx}{1}}{\frac{siinx}{cosx}} = sinx + \frac{cosx}{1}\frac{cosx}{sinx} = \frac{sinx}{1} + \frac{cos^2x}{sinx} = \frac{sin^2x}{sinx} + \frac{cos^2x}{sinx}$ So, is that $2sinx$? T: One third plus one third equals… what? S: Two thirds. T: Did the bottom change? S: No. In this excerpt, the student does not realise that if you add (or subtract) two fractions with the same denominators then you add (or subtract) the numerators of each fraction, not the denominators. In this case, the teacher notices the deficit in the students' mathematical thinking. His response to resolve the student struggle was what Teuscher et al. (2017) refer to as elaborating on the student's existing mathematical knowledge by giving the class an analogous question to solve. Warshauer (2014) refers to this type of assistance as *probing guidance.* In this regard the teacher asked *collecting information* questions.	

For DCAT 2 Activity 1 questions, no 'errors' under 'misconceptions and errors' were observed and was consequently omitted from Table 4.7.

In collaboration with the students, the teacher simplified the expression $\frac{sinx}{cos^2x} \div \frac{tanx}{cos^2x}$. The productive struggle type for $\frac{sinx}{cos^2x} \div \frac{tanx}{cos^2x}$ was thus labelled 'teacher illustrated'. Also, the simplification of the trigonometric expression $\frac{2cosx}{sinx} + \frac{2sinx}{cosx}$ was omitted from Table 4.7, since no productive struggles were observed for this activity question. The columns for the productive struggles 'Getting started' and 'Uncertainty in explaining & sense making' were also omitted from Table 4.7, since these productive struggles were not observed during DCAT 2 Activity 1 questions.

Although Kilpatrick et al. (2001) state that *adaptive reasoning*- is the ability to think logically about the connections between concepts, Chapman (2015) posited that students often ignore the rules and processes that should be used to solve a question. When simplifying the expression $\frac{cos^2x}{1-sinx}$ students should have the *adaptive reasoning* to know that the only substitution that can be made is that of replacing cos^2x with $1 - sin^2x$. The expression in the denominator of $\frac{cos^2x}{1-sinx}$ is in already its simplest form. However, in this study very few students could simplify $\frac{cos^2x}{1-sinx}$. The teacher resolved this struggle (of not being able to recognise that cos^2x can be replaced by $1 - sin^2x$) by explaining to the class the relationship between $\frac{1-x^2}{1-x}$ and $\frac{1-sin^2x}{1-sinx}$. In their investigation into a diagnostic assessment for the *sine rule,* Andika, Juandi and Rosjanuardi (2017) also found that students lack *adaptive reasoning.* Wilson and Heid (2010) noted that *adaptive reasoning* requires students to recognise existing assumptions, adjust to current assumptions by comparing assumptions and conventions, and work in different mathematical contexts.

In DCAT 2 Activity 1 questions, the students continued to struggle with some of the same concepts as they did in the DCAT 1 activities – for example, manipulation of fractions and cancellation errors. The number of 'getting started' productive struggles diminished between the DCAT 1 activities and DCAT 2 Activity 1 questions; but students still appeared to struggle with 'carrying out a process', for example in factorisation and cancelling terms. Mensah (2017) contended that student errors in trigonometry are based on the students' weakness at executing arithmetic operations. Some students struggled with operations on algebraic fractions that involved trigonometric functions. In attempting to simplify a trigonometric expression, some struggling students made cancellation errors. In his analysis of errors made by students required to find trigonometric derivatives, Siyepu (2015) also found that students were prone to committing cancellation errors. Moreover, when they had to factorise, some students expressed beliefs that were misconceptions. In their investigation of the difficulties that students experience with factorisation questions, Burhanzade and Aygör (2015) stated that some students could not factorise because they did not know trigonometric identities. But in this study, the students were given the three trigonometric identities; so their inability to factorise may have stemmed from a lack of prior knowledge.

The teacher resolved the DCAT 2 Activity 1 questions by explaining basic algebraic and arithmetic skills. As in Table 4.3 above, the teachers questioning led to what Lee, Tyson, Kim and Kim (2020) called "follow-up actions" to remedy the productive struggle.

4.2.2.4 Illustration of students' productive struggles with DCAT 2 Activity 2 questions

This section analyses examples of students' productive struggles during DCAT 2 Activity 2 questions. Table 4.8 below shows examples and the frequency of their occurrence during DCAT 2 Activity 2 questions.

Table 4.8: Distribution of students' productive struggles with DCAT 2 Activity 2 questions

Questions	**Types of productive struggles**				
				Expressing misconceptions and errors	
Simplify each of these expressions	**Getting started**	**Carrying out a process**	**Uncertainty in explaining and sense making**	**Misconceptions**	**Total**
$\frac{1}{sin^2x} + \frac{1}{cos^2x}$	0	1	0	0	**1**
$\frac{sin^2x}{1 + cosx}$	0	2	1	1	**4**
$\frac{1}{tan^2x + 1}$	1*	1	0	1	**3**

$\frac{sin^2x + 3sinx + 2}{sin^2x + 4sinx + 3}$	2	2	1	1	**6**
Total	**3**	**5**	**2**	**4**	**14**

1* indicates that students reached an impasse after getting started.

The simplification of the trigonometric expression $sinxcosxtanx + cos^2x$ was omitted from Table 4.8, since no productive struggles were observed for this activity question. Also, the productive struggle 'errors' were omitted since none were observed during DCAT 2 Activity 2 questions.

During DCAT 2 Activity 2 questions, the simplification of the expressions $\frac{sin^2x+3sinx+2}{sin^2x+4sinx+3}$ yielded the most examples of students' productive struggles. In attempting to simplify the expression $\frac{sin^2x+3sinx+2}{sin^2x+4sinx+3}$, students struggled an equal amount with 'getting started' and 'carrying out a process'. The productive struggle type that occurred the most during Activity 2 for DCAT 2 level questions was 'carrying out a process'.

4.2.2.5 An analysis of the question with the highest number of productive struggles from DCAT 2 Activity 2

This section analyses the simplification of the expression $\frac{sin^2x+3sinx+2}{sin^2x+4sinx+3}$.

<u>Getting started</u>

S: I don't know where to start

T: What did you do? Talk with me…

The teacher used *general interpreting to* resolve this productive struggle, by encouraging the struggling student to suggest how he would start to simplify the expression $\frac{sin^2x+3sinx+2}{sin^2x+4sinx+3}$. To resolve this struggle, the teacher used *collecting information* questions. The response "What did you do? Talk with me…" from the teacher could be considered *vague* or *unfocused.* However, the teacher was also trying to *collect information* from the students. Also, the lack of conceptual understanding of the concept prevented the student from starting the question.

<u>Carrying out a process</u>

T: Okay, so you have $sinx + cosx$ and $sinx + cosx$?

S: Yes sir. $\frac{sin^2x+3sinx+sin^2x+cos^2x+sin^2x+cos^2x}{sin^2x+4sinx+sin^2x+cos^2x+sin^2x+cos^2x+sin^2x+cos^2x} = \frac{3sin^2x+3sinx+2cos^2x}{4sin^2x+4sinx+2cos^2x}$

T: Any ideas?

The struggling student does not recognise the quadratic expression, or how to factorise it. This might point to a lack of *adaptive reasoning*, proposed by Kilpatrick et al. (2001). At this point, the teacher did not intervene with suggestions regarding how to recognise and factorise a quadratic; instead, by using a *generating discussion* question, the teacher asked the class to resolve the struggle, by saying: "Any ideas?" In this regard the response from the teacher is *vague* or *unfocused.*

Uncertainty in explaining and sense making.

C: At the very top, I did *[this]:* $(sinx\ +\ 2)(sinx + 1)$.

S: You mean, like…?

C: Oh… not factoring?

T: No, factoring is the right word.

Although one of the class members knew the correct answer, he was unsure of how to express it. That is, the student did not know how to articulate his solution, which refers to a lack of *strategic competence.* In this instance the teacher uses an *elaborated* response to affirm the student's correct use of terminology. His response is also considered to be *telling.*

Expressing misconception

S: Well, I did it the way she did it, and got stuck; and I thought of factoring. Now I'm stuck *[with* $\frac{sinx+2}{sinx+3}$*]*... I don't know what to do. I feel like if I change the 3 and 2 to sin^2x plus cos^2x...

it will be just a lot of work, and nothing will cancel.

Some students continued to be confused about the concept of 'simplifying'. The struggling student simplified the expression $\frac{sin^2x+3sinx+2}{sin^2x+4sinx+3}$ to the expression $\frac{sinx+2}{sinx+3}$. However, he did not realise that his answer was then in its simplest form; he wanted to continue, and subsequently made his solution ‘worse’. Khuzwayo (2019) found that students could not fully simplify trigonometric expressions; the findings of this study would suggest that students do not understand the concept of ‘simplifying’ itself, which might imply a lack of *conceptual understanding* of the concept.

Khuzwayo (2019) found that five out of eight students could not apply the appropriate algebraic operations when dealing with trigonometric identity questions that contained factorisation. Thus far, the findings of this study agree with Khuzwayo’s (2019) assertion that students lack the skill of applying prior knowledge. However, unlike in her study – where a student divided the terms in the numerator by the terms in the denominator – in this study, to simplify $\frac{sin^2x+3sinx+2}{sin^2x+4sinx+3}$, instead of factoring the numerator and the denominator the student replaced 2 by writing out $sin^2 + cos^2x$ out two times in the numerator and three times in the denominator (to replace 3), thus ending up with six terms in the numerator and eight terms in the denominator.

4.2.2.6 Illustration of students’ productive struggles from DCAT 2 Activity 2 questions

Table 4.9 below shows examples of students’ productive struggles (and evidence to support their categorisation) during DCAT 2 Activity 2 questions, as well as how the teacher and class responded to these struggles. The simplification of the expression $\frac{sin^2x+3sinx+2}{sin^2x+4sinx+3}$ is excluded, as it was analysed in the previous section.

Table 4.9: Summary of examples of students' productive struggles from DCAT 2 Activity 2 questions

Questions	**Types of productive struggle**			
				Expressing misconceptions and errors
Simplify each of these expressions	**Getting started**	**Carrying out a process**	**Uncertainty in explaining & sense making**	**Misconceptions**
$\frac{1}{sin^2x} + \frac{1}{cos^2x}$	No struggles observed.	S: *[writes]* $\frac{sin^2x+cos^2x}{sin^2x} + \frac{sin^2x+cos^2x}{cos^2x} =$ $\left(\frac{cos^2x}{cos^2x}\right)\frac{sin^2x+cos^2x}{sin^2x} + \frac{sin^2x+cos^2x}{cos^2x}\left(\frac{sin^2x}{sin^2x}\right)$ The student was unsure of how to add algebraic fractions. The teacher asks if replacing 1 with $sin^2x + cos^2x$ would help; the student replies that she doesn't know. However, the teacher reduces the cognitive demand on the student by *telling* the student: "The second part that you did helps, so switch that back *[*$sin^2x + cos^2x$*]* to 1".	No struggles observed.	No struggles observed.
$\frac{sin^2x}{1 + cosx}$	No struggles observed.	C: To make it the same as the bottom, set sin^2x = 1 – cos^2x. *S*: *[baffled]* Say that again? During this exchange between a struggling student at the whiteboard and the class, the student fails to recognise that $sin^2x + cos^2x = 1$ implies that $sin^2x = 1 - cos^2x$. The class resolved this struggle by saying: "To make it the same as the bottom, you take the first equation sin^2x +cos^2x = 1 and subtract cos^2x, so you have sin^2x = 1 – cos^2x".	S: *[writes]* $\frac{sinxsinx}{sin^2x+cos^2x+cosx}$ T: I would advise against that. C: Against that? The teacher lowered the cognitive demand on the student by *telling* the student: "I would advise against that."	S: *[writes]* $\frac{1-cos^2x}{1+cosx} =$ $\frac{1-cosxcosx}{?}$ T: Hold up. On the right, if you had $x^2 -$ 9 and we were going to factor, right, it would be $(x + 3)(x - 3)$. That's what

				they are trying to tell you with this, except that we have 1 and cosine. T: Does it make sense? To resolve this struggle the teacher used *directed guidance,* by using an analogous algebraic question on the board for the students to see the similarity. In order to emphasise the analogy the teacher asked a *linking* and *applying* question.

$\frac{1}{tan^2x + 1}$	T: So, we're stuck? Although the students did not struggle to 'get started', all attempts to simplify yielded no result. The teacher did not intervene with an answer. To get a sense of whether any student was struggling, the teacher asked a *collecting information* question. This type of response from the teacher is *vague.*	S: Yeah, like the second part you make the bottom [bottom] cos^2x then you got to add it to the top too and add it across. In this incident the student is confused about the process of dividing two fractions.	No struggles observed.	C: Take the square root of the whole thing In this incidence, one of the class members wants to take the square root of .$\frac{1}{tan^2x+1}$

The students did not exhibit any productive struggles when they simplified the expression $sinxcosxtanx + cos^2x$. Also, no 'errors' as a productive struggle were observed during this activity, which was thus omitted from the table.

During DCAT 2 Activity 2 questions, some students continued to make mistakes with operations on algebraic fractions that had trigonometric expressions. Mhakure et al. (2014) contended that deficiencies in working with (arithmetic) fractions will hinder progress in algebraic fractions. Such deficiencies in operations with fractions might suggest why the students in this study still struggled with algebraic fractions in trigonometry.

The results of this study agree with Siyepu's (2015) finding that students showed poor understanding of fractional trigonometric expressions. In the simplification of the expression $\frac{sin^2x}{1+cosx}$, the teacher resolved one struggle by giving the students a similar question. The class helped a struggling student to understand how to manipulate a trigonometric identity from $sin^2x + cos^2x = 1$ to $sin^2x = 1 - cos^2x$.

Next, the simplification of the expression $\frac{sin^2x}{1+cosx}$ yielded the second-highest number of examples of productive struggle. To simplify $\frac{sin^2x}{1+cosx}$, students struggled primarily to 'carry out a process', followed by 'uncertainty in sense making' and 'expressing misconceptions'. The students failed to see the relationship between sin^2x and $1 - cos^2x$ and consequently $1 + cosx$. As seen in the previous sections, the students exhibited a lack of adaptive reasoning.

4.2.2.7 Using the Newman Error Analysis to analyse students' mathematical thinking on assessment 2

In this section the NEA was used to establish whether there was a link or not between the errors that students made when they simplified trigonometric expressions on the second assessments, and the errors students made during their productive struggles in real time in DCAT 2 activities.

Table 4.10 below presents all the errors made by students during the second assessment. The number in each cell indicates the number of times each type of error was made per question.

Table 4.10: Distribution of NEA types among assessment 2 questions

Assessment question	**Types of NEA**				
Simplify the trigonometric expression	Comprehension	Transformation	Process skills	Encoding	Total
$1 + tan^2x$	0	2	6	3	**11**
$tan^2x - tan^2sin^2x$	0	4	5	4	**13**
$sin^2x + \frac{cosxsinx}{tanx}$	0	1	5	2	**8**
$\frac{1}{tanx} + \frac{1}{sinx}$	0	3	6	3	**12**
$\frac{cos^2x}{sin^2x}\frac{tan^2x}{1}$	1	5	11	2	**19**
$\frac{2sin^2x + 5sinx - 3}{3 + sinx}$	11	5	11	0	**27**
$\frac{sin^2x - 1}{sin^2x + sinx - 2}$	1	7	1	0	**9**
Total	**13**	**27**	**50**	**14**	**104**

For assessment 2, no errors at the *other hierarchy* were recorded, and this was therefore omitted from Table 4.10.

During Assessment 2 the students made the most errors in the *process skill hierarchy*, accounting for 50 out of 104 errors. Most errors in the *process skill hierarchy* were made

when the students attempted to simplify the expression $\frac{2sin^2x+5sinx-3}{3+sinx}$. Errors in the *process skill hierarchy* included the students' inability to recognise that the numerator is a quadratic, and to factorise the quadratic. As with what was seen in the in-class activities, the students struggled with the concept of factorisation. The findings of this study thus agree with Septiandi et al. (2020), who argued that students' lack of understanding of prior concepts prevents them from understanding advanced forms of similar concepts.
Furthermore, Chua (2017) contended that students who have been taught factorisation of quadratic trinomials in a procedural manner have been deprived of discovering or creating new mathematical knowledge such as applying a quadratic expression in a different context.

The second-greatest number of errors was made in the *transformation* hierarchy, accounting for 27 of 104 errors. Most errors in the *transformation* hierarchy were made when the students attempted to simplify the expression $\frac{sin^2x-1}{sin^2x+sinx-2}$.

Notably, there were no *reading errors* made on the assessment. Also, the students made the least number of errors when they attempted to simplify the expression $sin^2x + \frac{cosxsinx}{tanx}$.
When attempting simplify the expression $\frac{2sin^2x+5sinx-3}{3+sinx}$, students made the same number of errors in the *comprehension hierarchy* as in the *process skills hierarchy.*
The simplification of the expression $\frac{cos^2x}{sin^2x}\frac{tan^2x}{1}$ yielded the second-greatest number of errors by the students. In the *process skills hierarchy, t*he students made the same number of errors in their attempts to simplify $\frac{2sin^2x+5sinx-3}{3+sinx}$ as in their attempts to simplify $\frac{cos^2x}{sin^2x}\frac{tan^2x}{1}$.
Most cumulative errors were made when the students were asked to simplify the expression $\frac{2sin^2x+5sinx-3}{3+sinx}$.

4.2.2.8 Findings from the focus group interview on DCAT 2 assessment and activities

The purpose of this focus group interview was to get a deeper understanding of the mathematical thinking that students exhibited when committing the errors on the second assessment, and how these errors differ from the productive struggles during DCAT 2 learning activities.

The focus group met during their lunch break to discuss Assessment 2. Five members of the focus group were present: Brigitte, Aud, Maud, Amy and Pam. Each member of the focus group received their Assessment 2 scripts back to reflect on their answer choices.
Assessment 2 can be found in Appendix 7. Themes emanating from the focus group interviews were analysed. In the following interview excerpt, R = the researcher.

Basic algebraic operations

The focus group interviews revealed some deep-seated misconceptions about algebraic operations, particularly the addition and subtraction of algebraic fractions and the factorisation of trigonometric expressions. For example, for the question shown in Figure 4.5 below, Brigitte did not recognise the difference between two-squares factorisation in the numerator; neither did she recognise the quadratic expression in the denominator.

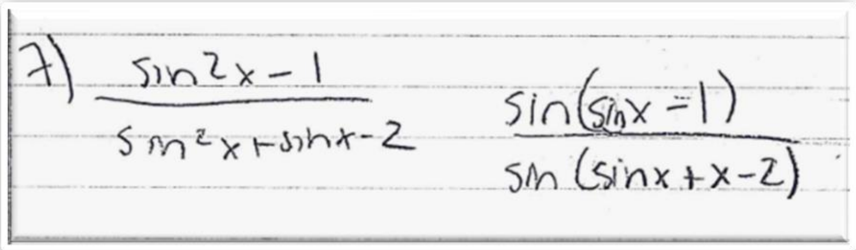

Figure 4.5: Brigitte's response to Question 7 on the assessment

Brigitte attempted the same incorrect factorisation method for both the denominator and the numerator.
Khuzwayo (2019) observed similar errors in her investigation into students' understanding of trigonometric identities. She stated that:

> "the learners' lack of understanding of trigonometric identities is rooted in a number of mathematics topics – this includes their lack of knowledge in algebra, such as factorising; fractions; working with like and unlike terms; etc. – and trigonometric ratios." (2019, p.6).

The following interview excerpt illustrates evidence of Brigitte's problem:

Brigitte: I got that one wrong *[question 7]* because I didn't… I didn't think about factoring. So I just didn't factor anything, and then

	I understand what you did as we went on and learned you should factor it and it would be easier. But at this point...
Aud:	After you explained *[wrote]* it out, you try to multiply and add everything. *[Aud refers to the fact that to check factorisation, one must multiply the factors to see if the result is the original question.]*
Brigitte:	Because it could work out. I didn't know that you could factor it.
R:	I think that you underestimate yourself.
Aud:	Thank you!
R:	I really think that you underestimate yourself.
Brigitte:	I really don't like factoring!
R:	Maud? What do you say?
Maud:	I got it right.
R:	Impressive, Maud. Number 7 *[question 7]*?
Brigitte:	I got minuses instead of plusses, and I don't know how to factor. Okay, I factored *that* one. *[referring to* $\frac{sin^2x-1}{sin^2x+sinx-2}$*]*
R:	Difference between two squares. Denominator or numerator?
Brigitte:	I didn't do *[attempt]* the top, the numerator. I didn't do *(sinx+1)(sinx-1)*. I did *sinx(sinx-1),* then I did $sin^2x - sinx - 2$, so I didn't factor it. I might have factored at number 6 *[question 6]*. I did, but got minuses instead of plusses. So I got there; it's just, I got the wrong signs. I don't know why I factored.

In Chapter Three, Ravitch (2010) noted that federal policymakers in the United States traditionally quantify teacher and student performance using numbers and statistics. Few would disagree that most mathematics teachers in the United States generally teach a topic with the emphasis on students passing a test, and not necessarily for long-term retention. At the beginning of the excerpt above, Brigitte noted that she "didn't think about factoring", an operation that comes naturally for a mathematician. Brigitte's failure to 'think about factorisation' is what Chevallard (2010) argues happens to mathematics as a discipline through didactic transposition, when the discipline depends on institutional settings. In Brigitte's case, the institutional setting is the teachers' eagerness for her to obtain high test scores in the short term, rather than long-term retention of knowledge and lifelong acquisition of mathematical skills. In this regard, Bosch et al. (2021) contended that the notion of didactic transposition indicates that learning and teaching practices do not start when the teacher and the student meet in the classroom. However, the starting point of didactic transposition originates from the selection and elaboration of *knowledge to be taught* from so-called *scholarly knowledge.* Bosch et al. (2021) argued that three institutions intervene in this didactic transposition – the scholarly institution of knowledge, the school institution where the knowledge has to be transposed, and the *noosphere,* which is defined as the parties involved that decide the teaching processes.

Over-reliance on $\boldsymbol{sin^2x + cos^2x}$

Members of the focus group continued to struggle with replacing 1 with $sin^2x + cos^2x$. In Figure 4.6 below, Maud superfluously replaces 1 with $sin^2x + cos^2x$, although she correctly replaces $tan^2x\ with \frac{sin^2x}{cos^2x}$; she fails to recognise that in order to add the 1 to $\frac{sin^2x}{cos^2x}$, the most appropriate substitution for 1 would have been $\frac{cos^2x}{cos^2x}$.

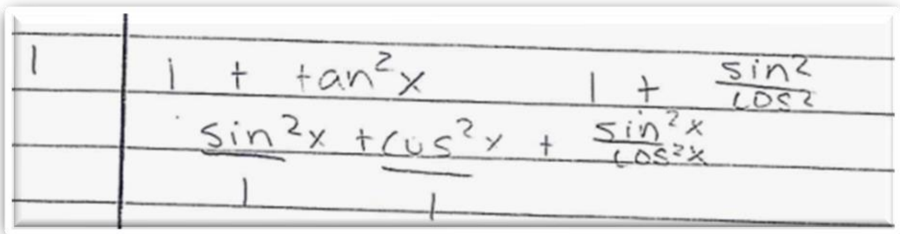

Figure 4.6: Maud's response to Question 1 on the assessment

Koyunkaya (2016) reported that students struggled when mathematical topics were integrated, and suggested that students' inability to transfer mathematical knowledge might be because students are only comfortable when working on routine procedural questions. Also, the findings from the focus group interview agreed with Rohmah and Ekawati (2018), who asserted that students usually have difficulty in manipulating formulae in trigonometry. The following interview excerpt shows evidence of Maud's problem.

R: Why did you do that? What was that you were saying earlier on? Let me just speak to Maud. You said you changed the 1 to $sin^2x + cos^2x$; is that correct?

Maud: Yes, instead of doing $\frac{cos^2x}{cos^2x}$. *[Maud means that she should have replaced 1 with* $\frac{cos^2x}{cos^2x}$ *to simplify* $1+tan^2x$*.]*

R: So I'm trying to figure out, where does it come from? *[replacing with* $sin^2x + cos^2x$ *]* Is it because that's the only thing you knew, like Brigitte said earlier? You *[meaning Brigitte]* changed every 1 to $sin^2x + cos^2x$. What was the reasoning again, Brigitte?

Brigitte: We just used it a lot.

R: In the classroom?

Brigitte: Yes.

R: So, you just replicate that in the assessment.

Brigitte: Yeah… it's not like we used it for every problem; but every single time we saw 1, we changed it. And if we were up on the board, and we were wrong, as soon as we saw the 1, we changed it. But then… like, now we know we don't have to change it every single time. There are only certain times that you should change it. But by then, I was 'there is 1', and changed it.

R: Was it your reasoning, Maud?

Maud: Yeah. I just needed the 1 to be something else.

Brigitte: I feel like I can't do anything with the 1 when it is just 1.

The students did not seem to resolve the issue of the overusing of $sin^2x + cos^2x = 1$. In Chapter Two, Kilpatrick et al. (2001) referred to this indiscriminate use of $sin^2x + cos^2x = 1$ as having a lack of *procedural fluency.* In all observed cases, the replacement of $sin^2x + cos^2x = 1$ is not incorrect; however, the replacement is not *suitable.*

The meaning of 'simplification'

During the focus group interview it became clear that some students did not know when to 'stop' when simplifying trigonometric expressions. This means they did not have a cognitive understanding of the concept of simplification. Aud argued that the answer should be 'harder' than the question that she started with.

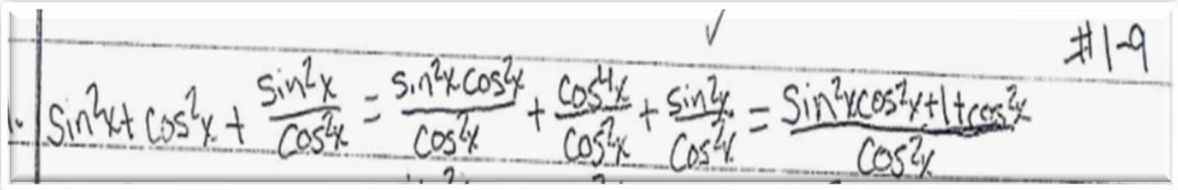

Figure 4.7: Aud's response to Question 2 on the assessment

In Figure 4.7 above, Aud superfluously replaced 1 with $sin^2x + cos^2x$. She then continued, to find a common denominator. All the steps that Aud followed to 'simplify' $1 + tan^2x$ were correct; however, the answer became more cumbersome than the original question.
Fu, Zhong and Zeng (2006) stated that it is difficult to define the simplification of trigonometric expressions clearly; for example, it is not clear which is 'simpler', sin^2x or *2sinx*. However, the issue of 'simplification' is not only bound to trigonometric expressions. In his investigation into the errors that students make when they calculate trigonometric derivatives, Siyepu (2015) found that students had little ability to simplify trigonometric functions. The following interview except provides evidence of Aud's problem.

Aud: I got the first part of the second step right – should have left a cosine underneath the sin^2x, but as earlier I decided to be expanded out.

R: So, why do you think, Aud… why do you think you don't know when to stop?

Aud: I guess I think the answer couldn't be that simple; there's… in my mind, there's no way it could have been that simple.

R: Why, do you think?

Aud: Because it looks like it's supposed to be harder.

R: Okay...

Brigitte: The problem looks really complicated, but the answer is really simple, and basically given to you. That's the only thing that messes with me.

During DCAT 2-level activities, the questions that exhibited the most productive struggles required factorisation for their simplification. The findings from the focus group interview suggest that some students had not resolved the problems they'd had with factorisation. These findings agree with Beukes (2015), who found that Mathematics Level 2 students struggle with factorisation, especially with the difference between two squares. 'Mathematics Level 2' refers to mathematics offered at second-year level to students who attend a South African Technical and Vocational Educational and Training college.) The findings from the focus group interviews also concur with Ngoveni and Mofolo-Mbokane (2019), who contended that students struggled with the factorisation of trinomials; in the case of this study, trigonometrical trinomials. These struggles with factorisation would explain why, during the second assessment, 49 out of 104 errors involved questions that required factorisation for simplification. Also, 36 out of 104 errors involved questions that featured trinomials.

Similarly to the outcome of the first focus group interview, the justification for using certain techniques to simplify trigonometric expressions was flawed. For instance, Maud used $sin^2x + cos^2x$ instead of $\frac{cos^2x}{cos^2x}$ in her simplification of the expression $1 + tan^2x$. In terms of ATD, the justification for the *technique* used to simplify the *type of task* is flawed. That means the *technology* under the knowledge block cannot be used to justify the *technique* under the practical block.

4.2.3 Data Analyses on DCAT 3 – Critical thinking level activities

This section presents an analysis of activities at the DCAT 3 level. As with DCAT 1 and 2, two activities were analysed at DCAT 3 level. The questions for the DCAT 3 level were considered to be the most difficult of the three levels. Thus, the students' progress and performance on DCAT 3-level activities may be noteworthy, compared to their progress and performance on the DCAT 1and 2 activities.

4.2.3.1 Illustration of the students' productive struggles on DCAT 3 Activity 1 questions

The students' productive struggles on DCAT 3 Activity 1 questions are summarised in Table 4.11 below, which shows examples of their struggles experienced when simplifying the trigonometric expressions in the left-most column.

Since the questions in the assessment were at the DCAT 3 level, the analysis gave an indication of the productive struggles experienced by the students during the most difficult activities.

Table 4.11: Distribution of types of students' productive struggles from DCAT 3 Activity 1 questions

Questions	Types of productive struggle					
				Express misconceptions and errors		
Simplify each of the following expressions	**Getting started**	**Carrying out a process**	**Uncertainty in explaining and sense making**	**Expressing misconcep tions**	**Errors**	**Total**
$tan^2x - tan^2xsin^2x$	1	1	0	0	0	**2**
$\frac{2sin^2x - sinx - 1}{sinx - 1}$	2	3	0	0	1	**5**
$tanx + \frac{cosx}{1 + sinx}$	1	3	0	0	0	**4**
$\frac{sinx - cosx}{sinx + cosx} + \frac{sinx + cosx}{sinx - cosx}$	0	3	1	2	0	**6**
Total	**4**	**10**	**1**	**2**	**1**	**18**

In DCAT 3 Activity 1 questions, most examples of students' productive struggles came from the simplification of the expression $\frac{sinx-cosx}{sinx+cosx}+\frac{sinx+cosx}{sinx-cosx}$. For this question, students struggled with 'carrying out a process' and 'expressing misconceptions'. The productive struggle type that occurred the most during DCAT 2 Activity 2 questions was 'carrying out a process'.

4.2.3.2 An analysis of the question from DCAT 3 Activity 1 that generated the highest number of productive struggles

The simplification of the expression $\frac{sinx-cosx}{sinx+cosx}+\frac{sinx+cosx}{sinx-cosx}$ resulted in the most examples of productive struggles – there were six student productive struggles observed on this question.

<u>Getting started</u>

No student had any difficulty 'getting started' with the simplification of the expression $\frac{sinx-cosx}{sinx+cosx}+\frac{sinx+cosx}{sinx-cosx}$.

<u>Carrying out a process</u>

S: *[writing]* $\frac{sinx-cosx}{sinx+cosx}+\frac{sinx+cosx(-1)}{sinx-cosx(-1)}=\frac{sinx-cosx}{sinx+cosx}+\frac{sinx-cosx}{sinx+cosx}$, just so that you have a similar denominator.

C: *[writing]* $sinx-cosx\frac{sinx-cosx}{sinx+cosx}+\frac{sinx-cosx}{sinx+cosx}(sinx+cosx)$.

I know that $sinx$ times $sinx$ is sin^2x…

T: Okay, so write it!

In this excerpt, one of the struggling students' classmates helped the student by explaining the correct process for finding a common denominator. Throughout this chapter the common theme has been the students' lack of understanding of basic algebraic processes. Khuzwayo (2019) commented that the simplification of trigonometric ratios posed difficulties for students; thus, the procedure for adding and subtracting trigonometric ratios should be constantly reviewed. This inability to perform basic processes and procedures in mathematics points to students' lack of *procedural fluency*, as suggested by Kilpatrick et al. (2001).

Uncertainty in explaining and sense making

C: That's what I started to do, but I didn't know how to do it, 'cause, uh… *sinx – cosx* times *sinx+cosx…* I don't know what that is. I don't know how to FOIL.

T: Okay – get up there, and we try again. *[the teacher encourages a student from the class to go to the board.]*

The teacher resolved this productive struggle by encouraging the struggling student to come to the board and "try again". The teacher then *guided* the struggling student on how the FOIL method works. The FOIL method (First, Outside, Inside and Last) is used to simplify the product of two binomials: 'First' indicates that the first term in one binomial should be multiplied by the first term in the other binomial; 'Outside' means the first term in the first binomial should be multiplied by the second term in the other binomial; 'Inside' indicates the second term in the first binomial should be multiplied by the first term in the second binomial; and 'Last' means that the second term in the first binomial should be multiplied by the second term in the second binomial. This type of response by the teacher is referred to as *directed guidance.* That means the teacher resolved the struggle based on the teacher's thinking. Again, the struggling student lacked *procedural fluency.*

Expressing misconceptions

S: What we said was that $sin^2x + cos^2x = 1$; we said that *sinx* + *cosx* = $\frac{1}{2}$.

T: False. Very, *very* false.

C: *[Laughter from the class]*
$\frac{sinx-cosx}{sinx+cosx}$; that is like adding one over one, but the top turns out to be negative one.
What? *[someone else in the class]*

T: False.

C: So… you shot down our answer?

T: Yeah. But I like the 'out the box' thinking. [But] you still have to stay in the realm of mathematics.

Although the teacher points out that the struggling student's assumption is false, he does not resolve the struggle by explaining why the assumption is false. He indirectly challenges the class to come up with different solutions by applauding the class for thinking out of the box. Since $sinx$ and $cosx$ represents real numbers, the struggling student fails to recognise that $\frac{sinx-cosx}{sinx+cosx}$ can represent a number, and that mathematical rules that apply to numbers may also apply to $\frac{sinx-cosx}{sinx+cosx}$. For example, if $sinx = \frac{1}{2}$ and $cosx = \frac{1}{4}$, then $\frac{sinx-cosx}{sinx+cosx} = \frac{\frac{1}{2}-\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}} \neq -1$, as the student claimed. This points to a lack of *adaptive reasoning* (Kilpatrick et al., 2001).

Next is a summary of and commentary on questions from DCAT 3 Activity 1 other than the simplification of $\frac{sinx-cosx}{sinx+cosx} + \frac{sinx+cosx}{sinx-cosx}$.

4.2.3.3 Examples of students' productive struggles from DCAT 3 Activity 1 questions

As was done for the DCAT 1 and DCAT 2 activities, Table 4.12 below shows examples of students' productive struggles during DCAT 3 Activity 1 questions, and the evidence to support their categorisation as well as how the teacher and class responded to these struggles. The simplification of $\frac{sinx-cosx}{sinx+cosx} + \frac{sinx+cosx}{sinx-cosx}$ generated most of the examples of students' productive struggles and was excluded in the summary, since this question has already been analysed.

Table 4.12: Summary of students' productive struggles from DCAT 3, Activity 1 questions

Questions	Types of productive struggle	
Simplify each of these expressions	**Getting started**	**Carrying out a process**
$tan^2x - tan^2xsin^2x$	S: $\frac{sin^2x}{cos^2x} - \frac{sin^2x}{cos^2x}\frac{sin^2x}{1} = \frac{sin^2x - sin^4x}{cos^2x}$ C: That is what I got. S: Is that it? C: *[mumbles, unsure of what to do]* T: That is not the final answer. The teacher let the class resolve this struggle by allowing them to discover that factorisation would be the most appropriate next step to simplify the trigonometric expression. He thus used *elaborated* responding by using the students' existing knowledge to discover the answer. And, as seen in previous sections of this study, the issue of 'simplification' does not seemed to be resolved. The teacher addressed this issue by simply saying: "That is not the final answer." This type of response by the teacher is known as *probing guidance* (Warshauer, 2014).	C: You can factor out a *sine square* at the top… Is that the next step? T: All right. S: *[writing]* $\frac{sin^2x}{cos^2x} - \frac{sin^2x}{}$… C: No! Just factor out! S: Right here? The struggling student did not use the process of factorisation. Here the class resolved the productive struggle.
$\frac{2sin^2x - sinx - 1}{sinx - 1}$	T: *[Student]* Can you put on the board what you got? S: I can, but it's not much. We did it one way and we thought that we were stuck, and we did it another way. The teacher resolved this struggle by allowing the struggling student to write her solution on the board, and to interact with the class to exchange ideas to resolve her struggles.	S: $\frac{2sin^2x - sinx - sin^2x + cos^2x}{sinx - sin^2x + cos^2x}$. You can't cancel, because they are subtraction and addition, right? C: Should it not be minus *cosine square* in the top, teacher? Or do we only change one? *[of them, meaning sin^2x or cos^2x]* T: No, we change both. So it would be minus, parenthesis, *sine squared x* plus *cosine squared x*. The teacher resolved this struggle by *telling* the students the answer, and in the process, lowered the cognitive demand on the students.

		This response from the teacher is referred to as *telling* by Warshauer (2014).
$tanx + \frac{cosx}{1 + sinx}$	S: $tanx + \frac{cosx}{1+sinx} = \frac{sinx}{cosx} + \frac{cosx}{1+sinx} = \frac{sinx+sin^2x}{cosx+cosxsinx} + \frac{cos^2x}{cosx+cosxsinx} = \frac{sinx+1}{cosx+sinxcosx}$ T: You're really close, keep going. Keep going! S: *[Confused]* I don't know what else to do. The struggling student has reached an impasse; the teacher encourages the struggling student to continue. This encouragement from the teacher is predicated by the teacher noticing that the student is close to simplifying the expression.	S: *[Writes]* $\frac{sinx+1}{cosx+sinxcosx}$. C: Well if I were you…*[muffled talking].* What about changing the 1 to, you know, would that benefit? C: *[Interrupting]* Can you not add a $sinx + 1$? T*: [Raising his voice]:* He has done nothing wrong; he is in a great spot. He's got to keep going! Although factorisation still seems to evade some students, the teacher *guides* the students by saying: "He has done nothing wrong; he is in a great spot. He's got to keep going!" In this instance, the class resolved the struggle.

There were no ‘Uncertainty in explaining and sense making’ or ‘Expressing misconceptions and errors’ productive struggles observed during DCAT 3 Activity 1 questions, and consequently these struggles were omitted from Table 4.12.

The predominant productive struggle in simplifying $\frac{2sin^2x-sinx-1}{sinx-1}$ was ‘carrying out a process’, followed by the productive struggle ‘getting started’. In simplifying $\frac{2sin^2x-sinx-1}{sinx-1}$, some students could still not comprehend under what circumstances they should replace 1 with $sin^2x + cos^2x$. One of the students wrote: $\frac{2sin^2x-sinx-sin^2x+cos^2x}{sinx-sin^2x+cos^2x}$ the numerator shows an error in the substitution. Thus, the findings of this study agree with Burhanzade and Aygör (2015) that understanding trigonometric identities is important for being able to factorise a trigonometric expression.

The interactions between the teacher, the struggling student and the class exemplify the triangular exchanges that occurred during the activities described in this chapter. This study thus agrees with one of the five practices formulated by Smith and Stein (2011) to promote effective discussion. They suggested that assisting the class to make mathematical connections is one practice that would advance effective discussion. Although the teacher *told* the struggling student what to do in certain circumstances, in Table 4.12 under ‘carrying out a process’ the teacher used one of the practices suggested by Smith and Stein (2011) to promote discussion in the classroom.

As in the previous DCAT activities, the students seemed to struggle with factorisation and basic operations on algebraic fractions.

4.2.3.4 Illustration of the students’ productive struggles on DCAT 3, Activity 2 questions

In this section, this study discusses the distribution of types of students’ productive struggles experienced while they attempted to answer DCAT 3-level Activity 2 questions.

The simplification of the trigonometric expression $\frac{2sinx+4}{3sin^2x+7sinx+2}$ yielded no productive struggles. Moreover, the productive struggles 'getting started' and 'expressing misconceptions and errors' were not observed during DCAT 3 Activity 2 questions.

During the second activity in the DCAT 3-level questioning, the simplification of the expression $\frac{sin^3x+cos^3x}{sinx+cosx}$ and that of $\frac{1}{\sin^2 x-1}+\frac{\sin x}{\sin^2 x-1}$ each yielded only one productive struggle. For this reason this study analysed both expressions.

Simplification of the expression $\frac{sin^3x+cos^3x}{sinx+cosx}$

Getting started:

The students did not have difficulty getting started. However, their approach to the solution of the question was flawed, as could be seen when they were *carrying out a process.* Such an inability to complete mathematical processes might indicate that the student has a lack of *procedural fluency.*

Carrying out a process

T:	Do we have an answer?
C:	One.
T:	Well? Put it up there!
C:	We basically divide it *[the denominator]* into the top *[numerator]* and got $sin^2x + cos^2x = 1$
T:	Ah! It's not that simple.

In this excerpt, the teacher anticipated that the students would struggle to 'carry out a process' by saying: "… I knew this was going to be a problem, because does anyone remember how to factor the sum or difference of two cubes?"

To address the struggling student, the teacher asked a *collecting information* question. The teacher then resolved this struggle by lowering the cognitive demand on the students by giving the students the formula for how to factorise the sum of two cubes and the difference between two cubes. The struggling student exhibited a lack of *procedural fluency.* This lack is linked to the students not having a *conceptual understanding* of mathematical processes.

Uncertainty in explaining and sense making.

T: …I knew this was going to be a problem, because… does anyone remember how to factor the sum or difference of two cubes?

S: *[Nodding]* No.

C: No. I don't remember

T: So, we have to factor the top *[numerator]*. Because once you factor the top *[numerator]*, it gets pretty easy. But I knew we *[you]* were not going to remember how to factor the sum of two cubes. So, the top factors… and I don't' know how to walk you through *[explain]* factoring it, so I will give you the factor *[how to factorise the sum or difference between two cubes]* formula.

T: *[Teacher writes on the board]:*

$$sin^3x + cos^3x = (sinx + cosx)(sin^2x - sinxcosx + cos^2x)$$

S: $\frac{sinx+cosx)(sin^2x-sinxcosx+cos^2x)}{(sinx+cosx)}$ = $sin^2x - sinxcosx + cos^2x$ = 1 - $sinxcosx$

In this exchange it was imperative for the teacher to give the formula for the sum of and difference between two cubes to the students. In this regard this study agrees with Wijnia et al. (2014), who contended that through direct instruction, correct procedures and knowledge are instilled, thereby eradicating any misconception. Consequently, the struggle was resolved by the student, after receiving the correct method for factoring the sum or difference between two cubes. The response from the teacher in this regard is referred to as *telling* by Warshauer (2014).

Simplification of the expression $\frac{1}{\sin^2x-1} + \frac{\sin x}{\sin^2x-1}$

For the simplification of $\frac{1}{\sin^2x-1} + \frac{\sin x}{\sin^2x-1}$ the students did not have any difficulty starting and (successfully) solving the question. However, the only productive struggle that was observed was the doubt that the class expressed about their answer.

Uncertainty in explaining and sense making

C: *[writing]* $\frac{1+sinx}{sin^2x-1} = \frac{1+sinx}{(sinx-1)(sinx+1)} = \frac{1}{sinx-1}$. It is not right.

T: No, it *is* right. Good job.

Although the struggling student wrote the correct answer, he/she still doubted whether the answer was correct. The struggle was easily resolved, by the teacher reaffirming that the student had written the correct answer. In this instance the teacher seemed obligated to tell the student that his answer was correct.

Expressing misconceptions

There were no issues regarding ‘expressing misconceptions’.

Over the DCAT 3 activities, ‘carrying out a process’, particularly operations on algebraic fractions and factorisation still seemed to be the predominant productive struggles of the students.

4.2.3.5 Comparisons between DCAT 1, DCAT 2 and DCAT 3 activities

To shed light on the first research question, that is, what type of productive struggles were observed during the simplification of trigonometric expressions, this study summarised and compared the total number of examples of types of productive struggles observed in DCAT 1, DCAT 2 and DCAT 3 activities.

Table 4.13: Summary of students’ productive struggles experienced in DCAT 1, DCAT 2 and DCAT 3 activity questions

Questions	Types of productive struggle					
				Expressing misconceptions and errors		
	Getting started	Carrying out a process	Uncertainty in explaining and sense making	Misconceptions	Errors	Total

DACT 1 Activity 1	4	5	3	2	1	**15**
DCAT 1 Activity 2	1	6	0	2	1	**10**
DCAT 1 Totals	**5**	**11**	**3**	**4**	**2**	**25**
DCAT 2 Activity 1	1	5	0	11	1	**18**
DCAT 2 Activity 2	3	5	2	4	0	**14**
DCAT 2 Totals	**4**	**10**	**2**	**15**	**1**	**32**
DCAT 3 Activity1	4	10	1	2	0	**17**
DCAT 3Activity 2	0	1	1	0	0	**2**
DCAT 3 Totals	**4**	**11**	**2**	**2**	**0**	**19**

Although the DCAT 2 activities were more difficult than the DCAT 1 activities, there was a decrease in the numbers of all but one type of productive struggle. The only increase was in the productive struggle ‘expressing misconceptions’. This study observed all of the misconceptions that the struggling students experienced in DCAT 2 Activity 2 questions related to weaknesses in factorisation and their inability to manipulate an expression in order to simplify it. This increase in the number of misconceptions suggests that some of the deep-seated misconceptions the students had in the DCAT 1 activities had not been resolved. However, the students could factorise algebraic expressions; but they could not see the similarities when they dealt with trigonometric expressions. This could also point to an inability to apply mathematical concepts in different mathematical situations. In this regard, the findings of this study agree with Khuzwayo (2019), who found similar results when investigating investigated trigonometric identities. However, Orhun (2015) stated that grave misconceptions can arise when students are introduced to new mathematical knowledge. This contention from Orhun (2015) is applicable to this study, since new mathematical knowledge (in terms of trigonometric expressions) was imparted. Moreover, the application of prior knowledge (manipulation of fractions) to trigonometry proved to be troublesome for some students. Consequently, this study suggests that the introduction of factorisation and the application of prior knowledge to trigonometry might explain the increase in the number of misconceptions.

Class intervention and teacher guidance contributed to resolve these misconceptions. Schoenfeld and Kilpatrick (2008) proposed that teachers should have broad content knowledge, so that they can impart different skills to their students. Statements such as: "I would advise against that" and "…that's what they are trying to tell you "Indicate the in-depth knowledge of the teacher. In this study, the teacher could ascertain what the struggling students' shortcomings were, and what would be the most effective way to address their struggles.

Furthermore, even though DCAT 3 Activity questions were more difficult than DCAT 2 and DCAT 1 activities, there was a decrease in the total number of productive struggles from DCAT 1 to DCAT 3 activities and from DCAT 2 to DCAT 3 activities. The numbers for 'carrying out a process' remained almost the same over the three DCAT activities.

Next, this study investigates the errors made by the students in the third assessment.

4.2.3.6 Using NEA to analyse students' mathematical thinking in the assessments

NEA was used to establish if there was a link between the errors made on assessment 3 and the errors that the students made during the DCAT 3 Activity questions.
Table 4.14 below shows the distribution of the errors made by the students when completing the third assessment.

Table 4.14: Distribution of NEA types among assessment 3 questions

Assessment question	**Type of NEA**				
Simplify the trigonometric expression	Comprehension	Transformation	Process skills	Encoding	Total
$sin^2x + cos^2xtan^2x$	7	0	0	3	**10**
$\frac{1}{tanx} + \frac{sinx}{1 + cosx}$	4	0	8	1	**13**
$\frac{1 + tanx}{1 - tanx} - \frac{1 - tanx}{1 + tanx}$	3	0	9	0	**12**
$\frac{2sinx + 1}{2sin^2x + 5sinx - 3}$	2	1	6	1	**10**
$\frac{cosx}{1 - cos^2x} + \frac{1}{1 - cos^2x}$	0	0	5	5	**10**
Total	**16**	**1**	**28**	**10**	**55**

The simplification of the trigonometric expression $\frac{1}{tanx} + \frac{sinx}{1+cosx}$ yielded the most errors made by the students during the third assessment. For the simplification of the trigonometric expression $\frac{1}{tanx} + \frac{sinx}{1+cosx}$, the students made the greatest number of errors at the *process skills hierarchy.*

In Assessment 3, most errors were made at the *process skill hierarchy*, accounting for 50.9 % or 28 of 55 errors in total.
These were the result of the students' inability to add algebraic fractions, and their lack of knowledge of the conditions under which terms may be cancelled.

4.2.3.7 Findings from the focus group interview on DCAT 3 activities and assessment

The purpose of this focus group interview was to get a deep understanding about the mathematical thinking which students exhibited when they committed errors on assessment 3, and how these errors differ from the productive struggles experienced during DCAT 3 learning activities.

The focus group met during their lunch break to discuss Assessment 3. Five members of the focus group were present: Brigitte, Aud, Maud, Amy and Pam. Each member of the focus group received their Assessment 3 scripts back to reflect on their answer choices. (Assessment 3 can be found in Appendix 8.) Themes emanating from the focus group interviews were analysed. In the following interview excerpts, R = the researcher.

Basic algebraic manipulations

Question 1 was designed to see if the students could correctly apply the PEMDAS rule in algebra. PEMDAS is the abbreviation for a rule applying the order of operations correctly. PEMDAS means parenthesis first, followed by exponents, division, multiplication, addition and lastly subtraction. In Figure 4.8 below, Maud erroneously grouped $sin^2x + cos^2x$, instead of first multiplying cos^2x by tan^2x and then adding the result to sin^2x. This type of error falls under the *process skills hierarchy* since Maud could not correctly apply the correct *procedure* to solve the question.

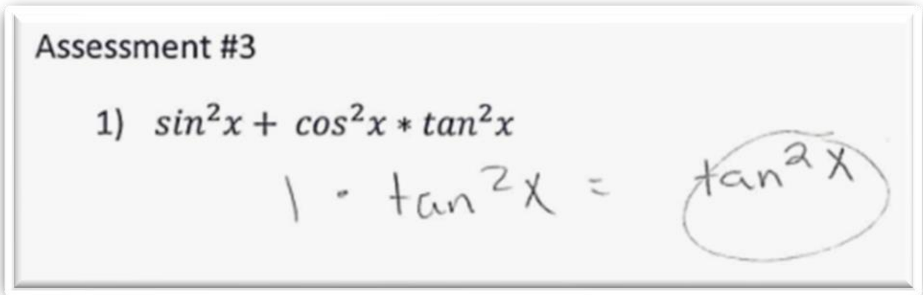

Figure 4.8: Maud’s response to Question 1 on the assessment

In his investigation into Grade 10 learners’ misconceptions and errors when simplifying algebraic expressions, Baidoo (2019) called this invalid use of principles a ‘concept error’, as illustrated by Maud’s response to Question 1. One of the strands of mathematical proficiency as discussed in Chapter Two is *procedural fluency.* Wilson and Heid (2011) posited that

procedural fluency refers to a student's ability to apply procedures *appropriately*. In Maud's case, the replacement of $sin^2x + cos^2x$ with 1 was not appropriate.
The following interview except is evidence of Maud's problem:

Brigitte:	The first one *[question 1]* I got right – I just didn't combine them back together *[i.e. simplified]*. I got $sin^2x + sin^2x$, but I didn't put the two together.
R:	You just left it as...
Brigitte:	Yeah, I just left it as sin^2x and *[plus]* sin^2x.
Aud:	$2sin^2x$; I got the right answer.
R:	And you, Maud?
Maud:	I got *[1+]* tan^2x.
R:	What did you do wrong, Maud?
Maud:	I changed it $[tan^2x]$ to $\frac{sin^2x}{cos^2x}$.
R:	Which is correct. Where else did you make a mistake?
Maud:	I would say I put $sin^2x + cos^2x = 1$ and changed *[added]* that to tan^2x. *[To simplify $sin^2x + cos^2x tan^2x$, Maud erroneously combined sin^2x and cos^2x to get 1]*

In Figure 4.9 below, Brigitte had difficulties with adding and subtracting rational *trigonometric* expressions. Although she added the two rational trigonometric expressions correctly, she did not cancel correctly. Brigitte's struggles with manipulating algebraic fractions are a snapshot of the errors that were committed in Assessment 3. During Assessment 3, 35 of the 55 total errors were committed when the students had to manipulate algebraic fractions involving trigonometric expressions.

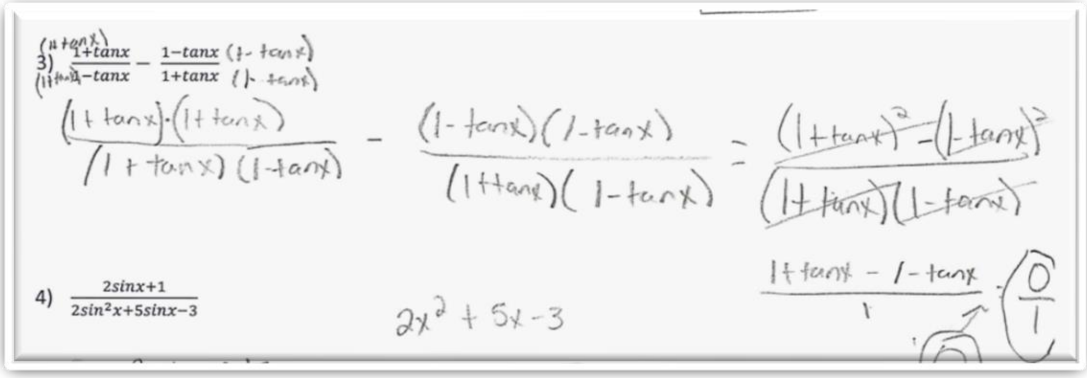

Figure 4.9: Brigitte's response to Question 3 on the assessment

Baidoo (2019) attributed comparable errors in the simplification of algebraic fractions to what he called 'mathematical language error'. A mathematical language error is an error made by a student because of a lack of understanding of technical mathematical language (Baidoo, 2019). Words such as 'simplify', 'factorise' and 'exponent' among others can be classified as mathematical technical language. The following interview excerpt provides evidence of Brigitte's problem.

R:	Number 3 *[simplification of* $\frac{1+tanx}{1-tanx} - \frac{1-tanx}{1+tanx}$*]*
Pam:	I got the bottom *[the denominator]* right, but instead *[in the numerator]…* I think I multiplied instead of adding them, because I got $4tanx$ + $2tan^2x$ instead of just $4tanx$. I don't know why I multiplied *[the tanx's]* instead of add *[just adding 2tanx + 2tanx]*.
R:	You multiplied the *2tanx* and the *2tanx* instead of adding it, and you got $4tan^2x$?
Aud:	Yes.
R:	And the numerator was okay?
Brigitte:	I didn't know; I multiplied to get the same thing *[the expression]* at the bottom *[the denominator]*. But then I combined, I put, like, $(1+tan^2x)(1- tan^2x)$, then at the bottom, I got *tanx(1-tanx)*, and then they cancelled. But… then you still had the *(1+ tanx)* minus the *(1-*

tanx) on top, from where I had them squared. So… [I] just had them totally wrong.

During the DCAT 3 level activities, the students also struggled with the manipulation of algebraic fractions – particularly in a question similar to Question 3 on the Assessment. Thus, the findings from the focus group interview would suggest that some struggles with algebraic fractions are still unresolved. Yang and Sianturi (2017) contended that trigonometry is a culmination of algebraic and geometric techniques; it stands to reason that a deficiency in either subject would hamper student success in trigonometry. According to the ATD, the imparting of these algebraic and geometric techniques lies clearly in the domain of the teacher. Thus, teachers should revisit their instruction methods to broaden students' understanding of basic algebraic operations.

The simplification of algebraic expressions

A lack of knowledge of certain basic algebraic manipulations was still hampering some students in the successful completion of the questions on the assessment. In Figure 4.10 below, we see that Brigitte still struggles with the concept of adding like terms.

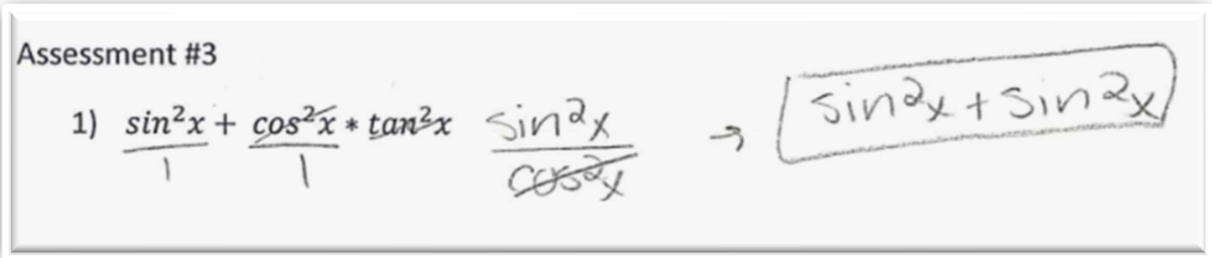

Figure 4.10: Brigitte's response to Question 1 on the assessment

According to Wilson and Heid (2011), *adaptive reasoning* (as discussed in Chapter Two) is the ability to recognise and adjust to assumptions and conventions. This ability also involves the comparison of different conventions and working in different mathematical contexts – in Brigitte's case, the ability to recognise that the simplification of $x^2 + x^2$ requires mathematically equivalent manipulations to simplify $sin^2x + sin^2x$. The following interview excerpt provides evidence of Brigitte's problem of a lack of adaptive reasoning.

Brigitte: The first one *[question 1]*, I got right; I just didn't combine them back together *[i.e. simplification]*; I got $sin^2x + sin^2x$, but I didn't put the two together.

R: You just left it as...

Brigitte: Yeah, I just left it as *sin*^{2}*x* and *sin*^{2}*x*.

Few would disagree that when confronted with a difficult question, students sometimes refer to a classroom moment in which the teacher explained a similar question. In Figure 4.11 below, Brigitte's response is an example of this.

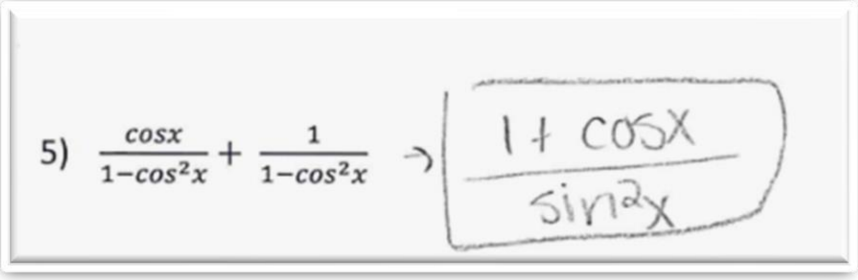

Figure 4.11: Brigitte's response to Question 5 on the assessment

Brigitte argued:

> "...probably *[I]* saw the cosines; and thought, okay, it's sin^2x, because we did that in practice. We had changed the 1-cos^2x to sin^2x. But now I can see you can just put them together and do the difference between two squares."

However, she does not realise that although she followed a similar approach to that used for a procedure for a question that was completed during one of the activities in the class, this procedure does not lead to the desired outcome for Question 5 in Figure 4.18. Kilpatrick et al. (2001) stated that students with *conceptual understanding* know more than isolated facts and procedures; rather, they have an integrated and functional understanding of mathematical ideas. In Brigitte's case, she did not follow a different path to simplify a trigonometric expression; rather, she relied on trusted methods that had been used in class. The following interview excerpt provides evidence of Brigitte's challenge.

R: Number 4 *[question number 4]*?

Aud: I forgot to put the 1 on top. *[referring to question 4]*

R: What did you put? *[referring to sin*^{2}*x + cos*^{2}*x]* ?

Aud: No, I just have *1- cosx.*

R: Oh, I see – that's your final answer.

Brigitte: I changed the bottom. *[the denominator]*

R: Which step in the answer?

Brigitte: On the second step, I changed the bottom to sin^2x, so that I got $\frac{1+cosx}{sin^2x}$.

R: Okay.

Brigitte: I didn't do the difference between two squares at the bottom; and then it cancelled out the ($cosx +1$)'s to get $\frac{1}{cosx-1}$.

R: Why do you think you did a thing like that?

Brigitte: I probably saw the cosines and thought, okay, it's sin^2x, because we'd done that in practice. We'd changed the $1-cos^2x$ to sin^2x; but now I can see you can just put them together and do the difference between two squares.

The last section deals with the highlights from the post-teaching meeting between the teacher and the researcher that this study conducted to get an overview from the teacher regarding DCAT 2 and 3 activities. R = researcher and T = Teacher.

4.2.3.8 Comparison between the number of types of errors for different assessments

This section gives a summary of the number of errors across the three DCAT-level assessments.

Table 4.15 below summarises the number of errors observed for the different assessments.

Table 4.15: Comparison between number of errors for different assessments

Assessment questions	**Type of Newman Error**				
Simplifying trigonometric expressions	Comprehension	Transformation	Process skills	Encoding	Total
Assessment 1	**5**	**6**	**23**	**17**	**51**

Assessment 2	**13**	**27**	**50**	**14**	**104**
Assessment 3	**16**	**1**	**28**	**10**	**55**
Total	**34**	**34**	**101**	**41**	**210**

For the first assessment, all the questions were attempted, so 'getting started' was not problematic. From Table 4.15, most errors occurred at the process skills and encoding hierarchies. The reader is reminded that for the process skills hierarchy the students should perform the necessary process to simplify the trigonometric expressions; and for the encoding hierarchy, the students are required to represent their answers in a mathematical way. Warshauer (2011) defined 'carrying out a process' as the ability to perform a mathematical process; in this case, to simplify a trigonometric expression flawlessly. It stands to reason that this productive struggle still seems to be unresolved.

For the second assessment, like the first assessment, all questions were attempted, so 'getting started' was not problematic. As for Assessment 1, most errors occurred at the process skills hierarchy, and consequently the productive struggle 'carrying out a process' seems to be unresolved. The second-highest number of errors occurred at the transformation hierarchy. (As a reminder to the reader, the transformation hierarchy is where the students should select an appropriate mathematical strategy to solve a problem.) On the other hand, the productive struggle 'misconception' was not defined by Warshauer (2011) as wrong thinking; however, it can be interpreted as an Indication of deep-seated misplaced ideas that are used to justify the process of finding a solution to a question. It stands to reason that this productive struggle seems to be unresolved.

For the third assessment, again all questions were attempted, so 'getting started' was not problematic. Most errors occurred at the 'process skill' hierarchy, which relates to the 'carrying out a process' productive struggle. The second-highest number of errors occurred at the 'comprehension' hierarchy.

4.2.3.9 Post-teaching meeting between the teacher and the researcher regarding DCAT 2 and 3 activities

During the post-teaching meeting between the teacher and the researcher regarding all the DCAT activities, the following themes became apparent. Whereas the post-teaching meeting described previously focused on DCAT 1 and 2, and thus on the teacher's view of the start of this study, this meeting focused on DCAT 2 and 3; thus, the focus of the meeting was on the latter part of this study.

The void of in-depth trigonometric knowledge

The teacher admitted that he needed to brush up on his trigonometry. However, for this study an in-depth knowledge of trigonometry was not required from the teacher. Only two trigonometric identities were discussed, namely- $sin^2x + cos^2x = 1, tanx = \frac{sinx}{cosx}$, and the fact that $sin^2x = sinx * sinx$. The following post-teaching meeting except provides evidence of the teacher's assertion.

R:	Today is May 20, 2019, and I'm discussing with the teacher what has transpired *[over]* the past eight or nine weeks. So, I just want to know what you have learned as a facilitator from the students' productive struggles in trigonometry.
T:	Definitely one thing that I should have known better at the beginning *[of the study was to]* brush up on my trigonometry, and use it, because I have *[not]* used it in a while – but I also think because of that I wasn't as confident, so I didn't give too much information to the students… which kinda helped.
R:	That was sort of a good thing.
T:	So, it made them work more, um… but as far as, like, in the actual classroom, I guess we struggled. And then you get the classic – that's good in theory, but it's not practical – of being okay in that space of… you present a problem, and there's no response.

Although Aslan-Tutak and Sarac (2017) found students of teachers with high trigonometry teaching efficacy scored higher on the trigonometry self-efficacy scale than students of teachers with low trigonometry teaching efficacy, in this study the teacher reported that because of his lack of in-depth knowledge of trigonometry, the students were made to work harder to solve the questions in the activities. Also, the teacher affirmed what became evident from the video observations: to resolve a productive struggle, a great deal of questioning was directed to the class.

Productive struggles' uncomfortable wait-time

The teacher noted that he had to overcome the 'uncomfortableness' of the silence that ensued when students did not respond to a question on the activities. This post-teaching meeting excerpt provides evidence of the teacher's experience of being uncomfortable during silence in the classroom.

R: Yeah, *[what about]* the awkwardness?

T: You want to fill that void…

R: I find that the most difficult thing to do, *[dealing with]* that awkwardness.

T: As awkward it is for you, it's worse for the students; and they also want to fill that void. And sooner or later one is going to speak up and do something; but just being okay with that awkwardness for a little while, as we went over on the eight or nine weeks, the students were more active to get up there *[students volunteered to go to the board to simplify trigonometric expressions].* And they were okay to get up there, not knowing everything, and going to the board, and most of them… There were still some of them that hated it, and shut down, but that… they would keep working, and keep searching for the right answer. So, that was kinda neat to see. *[The teacher noted that he was unsure whether the students pushed past the awkwardness of the silence in their quest to solve some of the questions in the activities, or if they attempted the questions to fill the void in the classroom]*

R: Was it beneficial to have taught the students before?

T: Right. Um… I think that was kinda the benefit of this class; it's the second year that I taught them. I do have a good relationship… We're now at the end of this, so it's been two of 14 or 15 students, it's been two straight years of math in my class *[except for two students, the teacher has been teaching the rest of the 15 students for two years]*. Um… we had a good relationship, and there was that trust there *[that]* I wasn't going to make fun of them, and *[I]* built that trust that the other kids aren't going to make fun of them; they're all kinda in the same boat, pushing in the same direction, being a bit vulnerable.

In their extension of the 'wait time' first used by Rowe (1986), Ingram and Elliott (2014) stated that: "silences are a powerful tool which can hugely influence the structure and nature of interaction" (p.10). However, this study noted – and the teacher alluded to this – that to extend the 'wait time', although "awkward" and "uncomfortable", can lead to better student outcomes regarding productive struggles. Students could fill that 'void' or silence in the classroom by going to the board, or giving an opinion about a question, since they trusted that the teacher would not criticise them unduly.

4.3 Summary

To struggle productively when simplifying trigonometric expressions is a multifaceted activity. Each tenet of NEA can be linked to a productive struggle type. During the DCAT 2-level activities on the simplification of trigonometric expressions, careful teacher intervention and support led to the successful resolution of productive struggles. However, this study observed that one of the main reasons students struggled with simplifying trigonometric expressions was the lack of deep understanding of basic algebraic concepts. Moreover, Weber (2008) contended that students who have difficulty with multiple representations of a concept also have difficulty with trigonometry.

Thus, this study agrees with Maglipong, Roble and Luna (2015) that instruction in these basic algebraic concepts must change in order for students to have a more complete understanding of the mechanics of mathematics, rather than just a method for them to remember processes. Koyunkaya (2016) contended in a literature review that trigonometry links algebraic concepts with geometry. It stands to reason that a lack of student skills in algebra would undermine success in trigonometry. Moreover, Jupri and Drijvers (2016) stated that among other issues in algebra, students have difficulty with the notation of variables. This difficulty could point to why some students struggle with *adaptive reasoning*, as was the case in this study.

In Chapter Five, this study focuses on the analysis of proving trigonometric identities. The analysis in Chapter Five follows the same pattern as Chapter Four. This similarity allows for ease of comparison in order to answer the research questions.

CHAPTER FIVE: DATA ANALYSIS – PROVING OF TRIGONOMETRIC IDENTITIES

5.1 Introduction

This chapter examines and analyses the data that was captured when a group of high-school students attempted to prove trigonometric identities at DCAT levels 1, 2 and 3. The analysis follows the same format as in the previous chapter. In Chapter Five, the focus is on how the students struggled productively with proving trigonometric expressions.
Again, the words *activity* and *activity questions* are used interchangeably. Also, this study uses the ‘=’ symbol instead of the equivalent symbol, as part of the notation.

As a reminder to the reader, this study will briefly refer to and show the interconnectedness of the preceding chapters with Chapter Five. In Chapter Three, this study justified the choice of a qualitative research methodology, and presented a case for why descriptive statistics could supplement this methodology. The qualitative methodology research, as described in Chapter Three, is used both in Chapter Four and in Chapter Five, to present a vivid description of the interaction between the struggling student and his or her classmates; the interaction between every struggling student and the teacher; the interaction between the teacher and the classmates; and lastly, the triangular interaction between teacher, classmates and each struggling student during an activity. Descriptive statistics are used to analyse the errors made during the assessments on the proving of trigonometric identities in this chapter.

5.2 Proving trigonometric identities

This section briefly discusses the proving of trigonometric identities as it relates to this study. Unlike Chapter Four’s data analysis, where the emphasis was on simplification of trigonometric expressions, this chapter focuses on proving trigonometric identities. A primary difference between the two processes is that in the simplification of a trigonometric expression, the students do not know what the ‘answer’ to the question should be; by contrast, in proving trigonometric identities the students must show that trigonometric functions on either side of the equivalent sign are equal. For example, to prove that *tanxsinx*

+ *cosx* is equivalent to $\frac{1}{cosx}$, the students can either show that if simplified correctly, *tanxsinx* + *cosx* yields $\frac{1}{cosx}$, or that $\frac{1}{cosx}$ can be written as *tanxsinx* + *cosx*.

5.2.1 Data analyses from DCAT levels

Next, this study explains the data analysis process that is followed for Chapter Five.
Chapter Five follows a similar process for data analysis to Chapter Four. However, while Chapter Four focused on the simplification of trigonometric expressions, Chapter Five places the emphasis on proving trigonometric identities.
For each DCAT level, a distribution of the productive struggle types encountered during the activity questions is presented. This is followed by a discussion of this distribution of the students' productive struggles.
Thereafter, the activity question that resulted in the most student productive struggles is discussed and analysed, and the remaining questions from the activity are summarised in a table and discussed.
The purpose of applying the NEA to the students' assessments in the activities is to establish whether or not there is a link between the errors that the students made when they proved trigonometric identities in the assessments, and the errors the students made during their productive struggles in real time in the activities.
This is followed by the distribution of the errors made by the students on each assessment, and a brief discussion of those errors. After that there is an account of the focus group interview conducted to corroborate or contradict the findings from the activities performed and the errors made in each assessment. In other words, the focus group interviews were conducted to deepen understanding of the nature of the NEA committed by students, and how they differ from the students' productive struggles in real-time activities.

Although other studies are also cited, on the whole Chapter Five also uses Boaler and Brodie's (2004) nine question types (discussed in Chapter Two) to categorise how the teacher responded to a student struggle in each case. The facilitating teacher was made aware of some of these question types at the beginning of the research process.

Next, a comparison is drawn between the productive struggles experienced by the students when proving trigonometric identities in the activities at each of the different DCAT levels.

After that this chapter compares and discusses the types of errors made by the students on the three assessments based on the activities for DCAT levels 1, 2 and 3. Lastly, highlights are presented from the post-teaching meeting between the teacher and researcher regarding DCAT 1, 2 and 3's activities.

Chapter Five concludes by highlighting the major findings of this chapter.

5.2.1.1 Data analyses from DCAT 1 – Basic cognitive abilities

The DCAT 1 Activity 1 questions for proving trigonometric identities are now analysed. For the DCAT 1 Activity 2 questions, the steps for proving a trigonometric identity were printed, cut out, scrambled and then handed to the students to unscramble and arrange in logical order. The teacher would then ask a student to present his or her steps on the board. Thus, the productive struggle was to see if the students could identify how to prove each trigonometric identity.

This activity required the students to investigate each step critically, and for each step, answer a question: "Why does this step follow the previous one?" From a teaching perspective, through this process the teacher could easily detect and rectify any flawed arguments from the students. This rectification could take the form of questioning, or if necessary, the reteaching of a mathematical process where a student's idea of it is incomplete.

5.2.1.1.1 Distribution of students' productive struggles on DCAT 1 Activity 1 questions

This section analyses DCAT 1 Activity 1-level questions. The analysis was done by observing and analysing the video recordings of the questions answered by the students in the classroom in real time.

For DCAT 1 Activity 1 questions, no productive struggles were observed for proving $cosxtanx = sinx$, $1 - sin^2x + cos^2x = 2cos^2x$, $\frac{\frac{1}{cosx}}{\frac{1}{sinx}} = tanx$ and $\frac{1}{sin^2x}\frac{1}{cos^2x} = \frac{1}{sin^2x} + \frac{1}{cos^2x}$.

Proving the trigonometric identity $16sin^2xcosx + 12cos^2xsinx = 4sinxcosx(4sinx + 3cosx)$ produced one productive struggle. For this question, the student expressed *uncertainty in explaining and sense making*. In this excerpt, T = teacher and S = student.

S: Teacher, can we make the right side equal to the left side, or does it have to be the left side equal to the right side – do you see what I'm saying? I expanded, and then simplified.

T: You can work backwards – that will be fine.

The teacher resolved this productive struggle by telling the student that it is acceptable to start on either side of the equals sign; although Osada and Supatmono (2020) did state that to prove a trigonometric identity, students usually start with the more difficult side of the identity and show that that side can be reduced to the easier side. For the question under discussion, however, it is difficult to discern which side the student considered to be 'more difficult'.

In this regard, the teacher's response was based on the student's mathematical thinking, so he used *telling* to resolve this struggle.

5.2.1.1.2 Analysis of students' productive struggles on DCAT 1 Activity 2 questions

This section deals with the analyses of DCAT 1 Activity 2 questions. Ten activity questions for DCAT 1 Activity 2 were presented to the students. The first five questions of the activity were posted in the normal way. This means the students were asked to prove, for example, $cosxtanx = sinx$. For the last five questions of the activity the students were asked to arrange a scrambled proof of each activity question in a logical manner. Similarly to the simplification of trigonometric expressions, the teacher asked a student to present their solution on the board. Question 6 of the activity questions is presented below. The rest of the activity questions can be found in Appendix 15.

<u>Question 6</u>

Prove that $tanx + \frac{2}{tanx} = \frac{sin^2x+2cos^2x}{sinxcosx}$.

$\frac{sin^2x}{sinxcosx}+\frac{2cos^2x}{sinxcosx}$	$tanx + \frac{2}{tanx}$	$\frac{sinx}{cosx} + 2\frac{cosx}{sinx}$

None of the students had any difficulties with DCAT 1 Activity 2 questions. Consequently, no productive struggles were observed for DCAT 1 Activity 2 questions. For questions 6 to 10, the students successfully arranged all the steps to the solution in each case. One of the main differences between proving trigonometric identities and simplifying trigonometric expressions is that with proving trigonometric identities, the solution is given. The 'how to' of getting to the solution is the challenge.

5.2.1.2 Using NEA to analyse students' mathematical thinking in Assessment 1

This study then used NEA to examine and analyse the types of errors made by the students on the assessment that followed the DCAT 1 activities. The reader is reminded that the purpose of applying the NEA to the students' assessments in the activities is to establish whether or not there is a link between the errors that the students made when they proved trigonometric identities in the assessments, and the errors they made during their productive struggles in real time in the activities.

5.2.1.2.1 Categorisation of the errors using NEA

This section deals with how this study categorises errors using NEA. Table 5.1 below gives a distribution of NEA types among Assessment 1 questions that were written by the students after DCAT 1 activities.

The number in each cell in Table 5.1 below indicates the frequency of observation of each type of error when the DCAT 1 assessment questions were analysed. For the assessment questions in Table 5.1, when students did not provide evidence and merely rewrote the trigonometric identity, it was coded as a *comprehension hierarchy* error. For example, a student might write $tanxsinx + cosx = \frac{1}{cosx}$, with no evidence to show how that conclusion had been reached.

Table 5.1: Distribution of NEA types among assessment 1 questions at DCAT 1 level.

Assessment question	Type of NEA				
Prove the following trigonometric identities	Comprehension	Transformation	Process skills	Encoding	**Total**
$\frac{tanx}{\frac{1}{cosx}} = sinx$	0	0	1	2	**3**
$\frac{\frac{1}{cosx}}{\frac{1}{sinx}} = tanx$	2	1	0	0	**3**
$tan^2xcos^2 = sin^2x$	3	0	1	1	**5**
$1+\frac{sinx}{cosx} = \frac{sinx+cosx}{cosx}$	4	1	1	2	**8**
$sin^2x + 2cos^2x = 1 + cos^2x$	1	1	2	0	**4**
$tanxsinx + cosx = \frac{1}{cosx}$	2	1	6	0	**9**
Total	**12**	**4**	**11**	**5**	**32**

During Assessment 1 the students made the most errors at the *comprehension* hierarchy, accounting for 12 out of 32 errors. Most errors at the *comprehension* hierarchy were made when the students attempted to prove that 1+ $\frac{sinx}{cosx} = \frac{sinx+cosx}{cosx}$. In Figure 5.1 below, the substitution of 1 with $sin^2x + cos^2x$ should have been done before the division sign. However, the student makes a comprehension error by injudiciously multiplying 1+$\frac{sinx}{cosx}$ by *tanx.*

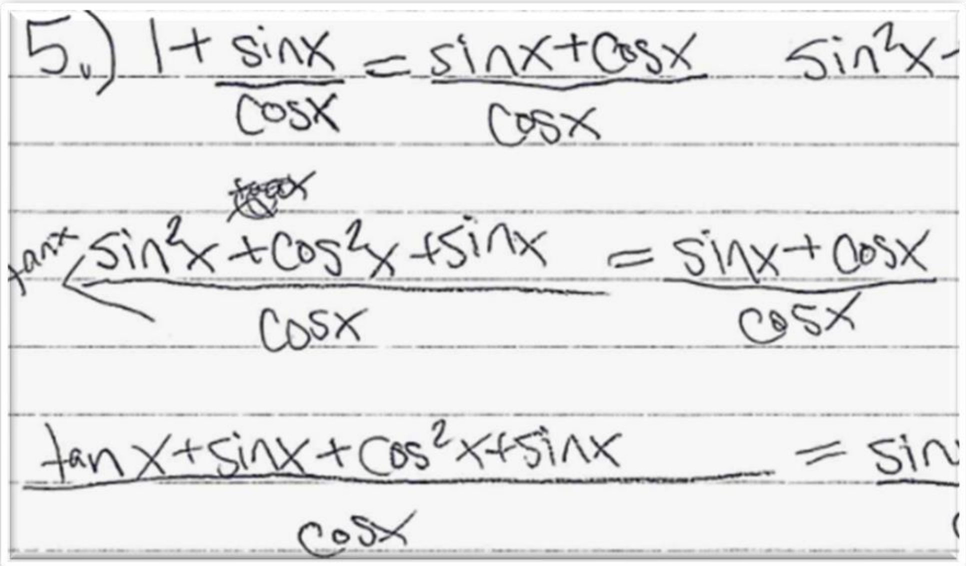

Figure 5.1: Error – Comprehension hierarchy

The second-greatest number of errors was at the *process skills* hierarchy, accounting for 11 out of 32 errors. Most errors at the *process skills hierarchy* were made when the students attempted to prove that $tanxsinx + cosx$ = $\frac{1}{cosx}$. This would suggest that the students still struggled with basic algebraic manipulations. In Figure 5.2 below, it is clear the student does not understand the process of adding two algebraic fractions.

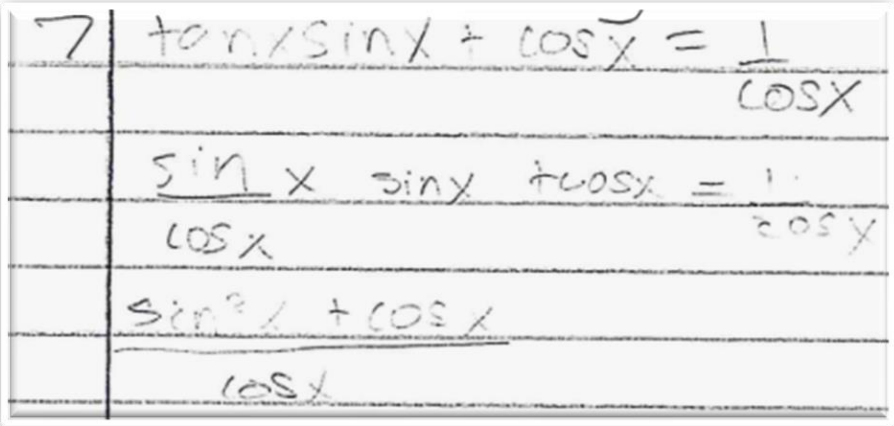

Figure 5.2: Er–or - Process skill hierarchy

Notably, there were no *reading errors* made on the assessment; and consequently, this NEA hierarchy was omitted from the table. Also, the students made the least number of errors when they attempted to prove that $\frac{tanx}{\frac{1}{cosx}} = sinx$ and when they attempted to prove that $\frac{\frac{1}{cosx}}{\frac{1}{sinx}} = tanx$

No NEA errors were observed for proving $tan^2x = \frac{sin^2x}{1-sin^2x}$, and consequently this activity question was excluded for analysis in Table 5.1 above. Also, no '*other'* NEA errors were observed for the first assessment, and this category was thus also excluded from Table 5.1. However, some students did use the trigonometric identity $sin^2x + cos^2x = 1$ effectively, as illustrated in Figure 5.3 below.

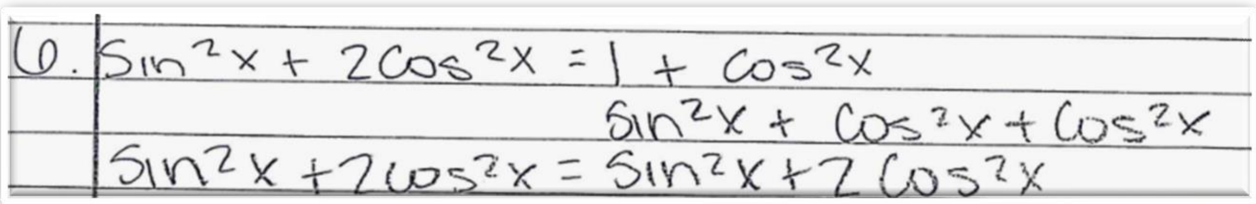

Figure 5.3: Effective use of $sin^2x + cos^2x = 1$

Furthermore, some students could readily 'dissolve' $2cos^2x$ as $cos^2x + cos^2x$, as illustrated in Figure 5.4 below.

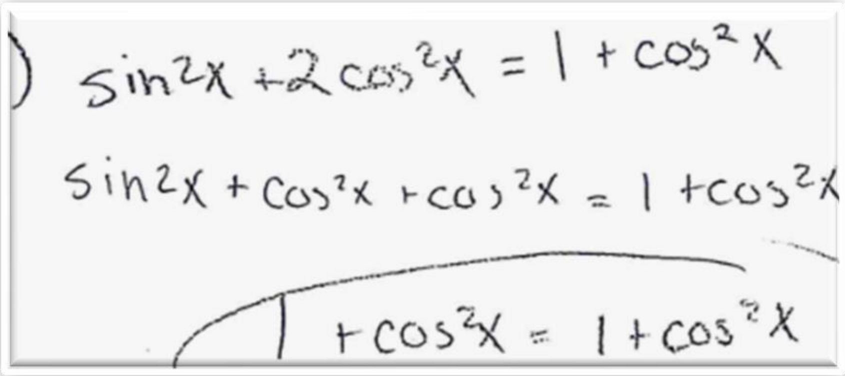

Figure 5.4: Dissolving of $2cos^2x$

In the next section, this study compares and analyses DCAT 1's Activity 1 and Activity 2 questions.

5.2.1.3 Discussion – DCAT 1 Activity 1 and Activity 2 questions

Only one productive struggle was observed during Activity 1 and Activity 2 questions on proving trigonometric identities; however, during the assessment, some of the same algebraic errors occurred that were observed in the simplification of trigonometric expressions (see Chapter Four). For example, the over- (and sometimes quite unnecessary) use of the trigonometric identity $sin^2x + cos^2x = 1$ still prevailed. In their investigation into whether problem-based learning can influence students' ability to prove trigonometric identities, Osada and Supatmono (2020) concluded that students' inability to perform basic algebraic operations, especially fractions, prevented them from proving trigonometric identities. Observing student performance in Activity 1 and Activity 2 questions, this study agrees with Osada and Supatmono (2020).

The focus group interview regarding the DCAT 1 assessment is now discussed.

5.2.1.4 Focus group interview – DCAT 1 assessment and activities

This section deals with the focus group interview which was conducted after the first assessment at the DCAT 1 level. The main purpose of the focus group interview was to obtain evidence as to whether students' productive struggles that were observed during DCAT 1 learning activities were resolved during the assessment of the DCAT 1 activity questions. Focus group interviews gave the researcher a deep understanding of the nature of the errors the students committed in proving trigonometric identities at the DCAT 1 level. Moreover, from the themes that came out of the interview, the researcher could determine whether the struggles observed from the video recordings of the DCAT 1-level activities had been resolved.

The focus group met during their lunch break to discuss Assessment 1. Four members were present: Aud, Maud, Amy and Pam. Each member had their Assessment 1 scripts returned to them so they could reflect on their answer choices.

Assessment 1 (regarding *proving trigonometric identities*) can be found in Appendix 9. Themes emanating from the focus group interview were analysed. In the following interview excerpt, R = researcher.

Over-reliance on replacing 1 with $sin^2x + cos^2x$

Two of the four members of the focus group struggled with Question 5.
Aud and Amy's statements indicating that they struggled with Question 5 on the assessment reaffirm this study's observation thus far that algebraic errors – specifically the addition of algebraic fractions, and over-reliance on the trigonometric identity $sin^2x + cos^2x = 1$ – still seemed to be preventing the students from successfully proving trigonometric identities. In Figure 5.5 below, Aud unnecessarily replaces 1 with $sin^2x + cos^2x$, which results in a more cumbersome expression. Furthermore, she gratuitously multiplies the resultant expression by $tanx$.

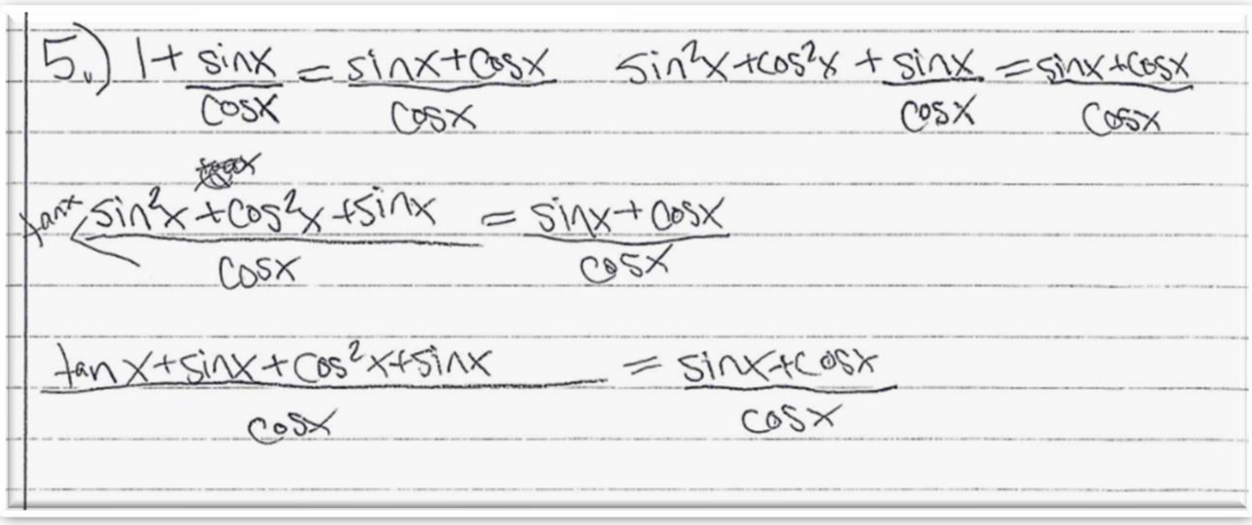

Figure 5.5: Aud's response to Question 5 on the assessment

Kilpatrick et al. (2001) referred to responses such as Aud's to Question 5 as lacking *procedural fluency*. This means Aud has not developed the skill to execute a mathematical process (in this case, adding algebraic fractions) in a suitable way.

The following interview excerpt provides evidence of Aud's over-reliance on $sin^2x + cos^2x$ = 1.

Aud: I don't know why I couldn't figure out number 5 *[Question 5]*.

R: Can you just read number 5?

Aud: 1+$\frac{sinx}{cosx} = \frac{sinx+cosx}{cosx}$

R: You can't remember why you struggled. But you struggled with number 5?

Aud: Yeah… I don't really know why. I probably tried to make the *sinx* + *cosx [equal to]* one, like, *[by]* squaring it. Um… or I may have tried to *[simplify]* 1+ $tanx$. I really don't remember what I did. I just kind of… struggled.

Manipulation of Algebraic Fractions

In Figure 5.6 below, Amy's response to Question 5 is baffling. She seemingly multiplied $\frac{sinx}{1}$ by $\frac{1}{cosx}$, and got $sinx + cosx$ as her answer. What is more confusing is that she then concluded that $1 + \frac{sinx}{cosx} = \frac{sinx+cosx}{cosx}$.

The excerpt below from the subsequent interview provides evidence of Amy's lack of understanding regarding adding algebraic fractions.

R: Amy, what did you struggle with?

Amy: Figuring out how to start; like with number 5 *[question 5]*. Figuring out how to get to, like, *that* part; 'cause then, if I couldn't get it on the left side to the right, I tried from the right and fix it to the left. *[Amy is* *having difficulty trying to prove the trigonometric identity, from left to right and vice versa.]*

R: Okay, number 5 *[question 5]* was problematic for you too?

Amy: Yeah.

Aud: Now that you say that, it makes sense; but I doubt that I did that.

Maud: I don't think that I did.

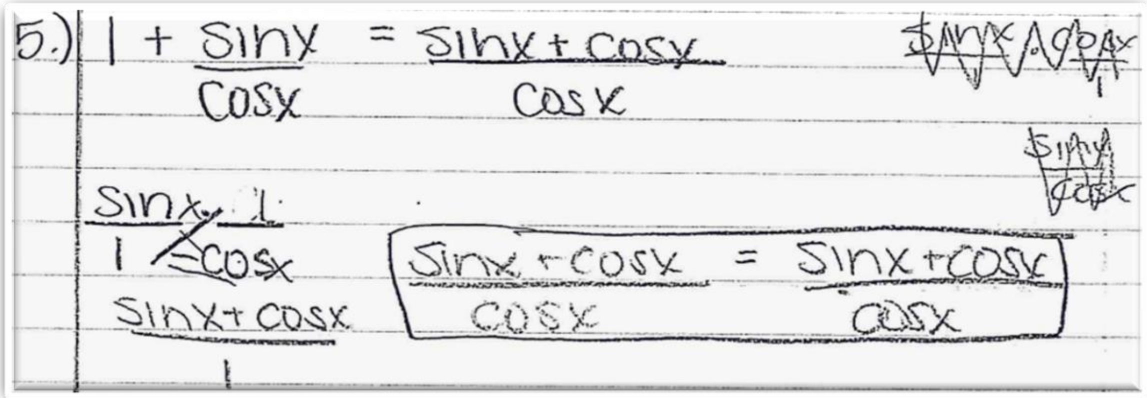

Figure 5.6: Amy's response to Question 5 on the assessment

This study concurs with the observation of Mhakure et al. (2014) that expertise in manipulating fractions does not necessarily assure success in algebra; but lack of understanding of the concepts of fractions will hinder progress in simplifying algebraic fractions – including trigonometric fractions.

On the other hand, Kilpatrick et al. (2001) referred to responses such as Amy's response to Question 5 as having a lack of *conceptual understanding* of how to add algebraic fractions.

5.2.2 Data Analyses on DCA– 2 - Application-level activities

This section presents an analysis of activity questions at DCAT 2 level. Similarly to the DCAT 1-level section, two activities (Activity 1 and Activity 2 questions) are analysed.

5.2.2.1 Distribution of students’ productive struggles on DCAT 2 Activity 1 questions

Next, a distribution of the productive struggles experienced by the students during DCAT 2 Activity 1 questions is presented.

Table 5.2 below summarises the distribution of productive struggles encountered in DCAT 2 Activity 1 questions.

Table 5.2. Distribution of students' productive struggles from DCAT 2, Activity 1 questions

Questions	Types of productive struggle				
				Expressing misconceptions and errors	
Prove the following trigonometric identities	**Getting started**	**Carrying out a process**	**Uncertainty in explaining and sense making**	**Misconceptions**	**Total**
$sin^4x - cos^4x = 2sin^2x$-1	1	1	1	2	**5**
$sin^3x + cos^3x = (sinx + cosx)(1 + sinxcosx)$	1	1	0	0	**2**
$(tan^2x + 1)(cos^2x - 1) = -tan^2x$	0	1	0	0	**1**
Total	**2**	**3**	**1**	**2**	**8**

The activity questions to prove $\frac{1}{tan^2x} + \frac{1}{tanx} = \frac{1+tanx}{tan^2x}$ and to prove $\frac{1}{sinx} - \frac{1}{tanx} = \frac{1-cosx}{sinx}$ were excluded from Table 5.2, since no productive struggles were observed for these activity questions. Also, no '*error'* productive struggles were observed for DCAT 2 Activity 1 questions, and thus this category was also excluded from Table 5.2.

During DCAT 2 Activity 1 questions, the students struggled the most when they were required to prove that $sin^4x - cos^4x = 2sin^2x$-1. For this question, the students struggled with '*expressing misconceptions'*. In this regard, the students struggled to identify how $sin^4x - cos^4x$ could be factorised.

The second-highest number of productive struggles occurred when the students were required to prove that $sin^3x + cos^3x = (sinx + cosx)(1 + sinxcosx)$. For this question, the students struggled equally between '*getting started'* and '*carrying out a process*'. These two productive struggles can be attributed either to the fact that the students did not know how to factorise $sin^3x + cos^3x$, or that they did not know how to use the distributive law to simplify $(sinx + cosx)(1 + sinxcosx)$.

Hourigan and Leavy's (2019) investigation into pre-service teachers' perceptions of learning revealed that conceptual understanding can be accomplished through multiple representations and making connections to other concepts. However, all the participants in this study were instructed in a procedural way.

5.2.2.2 An analysis of the question with the highest number of productive struggles from DCAT 2 Activity 1

This section analyses the activity question with the highest number of productive struggles. The students experienced the most productive struggles when they were asked to prove that $sin^4x - cos^4x = 2sin^2x - 1$.

The reader is reminded that this study uses the definitions of *getting started, carrying out a process, experiencing uncertainty in explaining and sense making* and *expressing misconceptions and errors* that were used by Warshauer (2011) when she investigated the productive struggles of a group of middle-school students in the mathematics classroom. In this excerpt from the video recording, T = teacher, S = struggling student and C = classmate/s.

<u>Getting started</u>

S: If I can take out squares x out of here… do I write squared x or just 2, so that I can get $sin^2x - cos^2x$?

T: How do you take out a square?

S: I don't know… $2(sin^2x - cos^2x)$?

T: Okay…

S: Can I do that?

T: …I don't think so. HOLD ON! Before you erase everything – how far did everyone else get?

C: That's it right there. We wrote the question down.

During this interaction, a student shows that he/she is struggling to 'get started' by asking the teacher a baffling question; how to get $sin^2x - cos^2x$. The teacher tried to resolve this struggle by asking a specific student in the class if he had any suggestions. Although the student did not resolve the struggle, he suggested starting the proof from the right-hand side. This suggestion from the student precipitated other ideas from the class, and thus resolved the 'getting started' struggle. The teacher seemed confused about the student's question about 'taking out a square'. In this regard the teacher adopted Aizikovitsh-Udi & Star's (2011) reasoning for why teachers should ask questions: to find out what is missing; facts, causes and reasons. This type of questioning is referred to as *probing* questions. Again, the teacher responded to the productive struggle based on the struggling student's mathematical thinking. Moreover, the teacher also wished to *generate a discussion* by asking: "How far did everyone else get?" In the excerpt above the struggling student exhibited a lack of conceptual understanding, since he was unsure of his own statement about 'taking out squares'.

Carrying out a process

T: Right; now that we see to the fourth power and to the fourth power, what are we thinking?

C: Take out a 2? *[The student erroneously means $2(sin^2x - 1)$]*

T: All right. If 'take out a 2' is the answer, what else could be there *[there be]*? *[The teacher infers that 2 cannot be a common factor to* $2sin^2x - 1$ *, since the expression only has one '2'.]*

S: …Take out a 4?

T: No, but a good guess.

C: I was thinking $(sin^2x - cos^2x)(sin^2x + cos^2x)$. But I don't know if that's the right format.

T: It absolutely is.

In this excerpt the teacher tries to resolve the struggle of 'carrying out a process' proposed by Warshauer (2011), to factorise the difference between two squares. In this instance he uses what Davis (1997) referred to as "interpretive listening". That means the questioning is based on investigating the students' mathematical knowledge, and importantly, how they think. Aizikovitsh-Udi & Star (2011) also referred to this type of questioning as a trait held by a *leveraging teacher* (see Chapter Two). This type of question, where the teacher focuses on an essential element (in this case, 'to the fourth power'), is called an *orienting and focusing* question. This type of question was immediately followed by a *probing* question, where the teacher required the struggling student to explain their thinking. This action by the teacher could not be indicated in the excerpt.

Eventually the struggle was resolved by one of the students in the class. The response by the teacher, where he uses student thinking, is referred to as *probing guidance* by Warshauer (2014). Some students still had an inadequate *conceptual understanding* of how factoring works in mathematics. The teacher made *equitable access* (Schoenfeld, 2018) to the activities less intimidating by giving a 'soft response' to an incorrect answer: "No, but a good guess."

<u>Uncertainty in explaining and sense making</u>

S: $(sin^2x - cos^2x)(sin^2x + cos^2x)$

T: All right. When we see the fourth power, all that is, *[is]* the difference in squares sin^2x and cos^2x , right? We just factored the difference of squares, and we're there. So, continue.

S: Now can I change the cos^2x to $1-sin^2x$?

T: Before you do that – what is $sin^2x + cos^2x$?

S: Oh! One… and $(sin^2x - cos^2x)$?

C: Is it not –he - $cos^2x = sin^2x-1$?

T: Uh-huh.

In this excerpt the student is unsure if his answer is correct. The teacher resolved this struggle by affirming the student's answer. Also, the teacher notices that the students are missing an 'obvious' answer (what is $sin^2x + cos^2x$), which then makes their path to a solution much easier. This type of response is referred to as *elaborated* response (see Chapter Two), where the teacher responds to a student's struggles by elaborating on the student's existing knowledge base. Regarding the first question in the excerpt, the teacher gave the struggling student some pertinent information to solve the question; however, he required the student to complete the rest of the solution. This type of question is referred to by Boaler and Brodie (2004) as *exploring mathematics meanings and relationships.* To further assist the struggling student, the teacher posed a *collecting information* question: "What is $sin^2x + cos^2x$?" The teacher asked this question to guide the students to reduce $(sin^2x - cos^2x)(sin^2x + cos^2x)$ to $(sin^2x - cos^2x) * 1$. Thereafter the class repla–ed - cos^2x with sin^2x-1 , without issue.

Expressing misconceptions

S: Could we, like, turn $cos^2x = 1 - sin^2x$ ... could we do $-1+sin^2x$ to the fourth to get $-cos^4x$? *[The student wants to know if* $-cos^4x = (-1 + sin^2)^4$*.]*

T: Don't go there yet.

S: Okay…

In this excerpt the struggling student 'expressed a misconception' about manipulating both sides of an equation. The teacher *attended* to the struggling student by observing that his or her struggle can be avoided by using an easier solution (in this case, factoring the difference between two squares). However, the teacher did not directly address the struggling student's misconception. This type of response by the teacher is referred to by Warshauer (2014) as *probing guidance*. The advantage of *probing guidance* is that the teacher guides the student to resolve a struggle based on the student's thoughts.

In their investigation of pre-service teachers' understanding of factorisation, Fitzmaurice and Hayes (2020) reported that although teachers need procedural and conceptual understanding if they are to teach for understanding, conceptual understanding should not be assumed, since a misconception in some dimension of understanding may lead to a misconception in another dimension of understanding. The point is that teaching factorisation should shift from being merely an instruction to 'factorise completely' to a broader perspective on how we can use previously acquired knowledge to solve problems in a new mathematical context.

5.2.2.3 Summary of the students' productive struggles from DCAT 2 Activity 1 questions

This section summarises the activity questions other than the one with the highest number of productive struggles from DCAT 2 Activity 1.

Table 5.3 below is a summary of the examples recorded of students' productive struggles during DCAT 2 Activity 1 questions, and the evidence that supports their classification as struggles, as well as how the teacher and class responded.

Table 5.3: Summary of the examples of students' productive struggles from DCAT 2 Activity 1 questions

Questions	Types of productive struggle	
Simplify each of these trigonometric expressions	**Getting started**	**Carrying out a process**
$sin^3x + cos^3x = (sinx + cosx)(1 + sinxcosx)$	C: I remember that it was really long, because I had to write the last one; but I don't remember how to do it. *[Student is referring to the factorisation of expressions with 'sums of two cubes'.]* T: The first set of parentheses is $(sinx + cosx)(sin^2x - sinxcosx + cos^2x)$ The teacher resolved this struggle by writing the formula for the sum of two cubes on the board. That is, the teacher resolved the struggle through *telling* the students the answer. No question was posed by the teacher.	C: So how do you get to that, the sum of the two cubes? T: The sum of two cubes? Do you remember Algebra 2? C: …No. Though the teacher explained the sum of two cubes formula to the students, they did not realise that $sin^2x + cos^2x = 1$. The teacher posed a *collecting information* question. In this instance the teacher *attended* to the students based on their *mathematical thinking.*

For DCAT 2 Activity 1 questions, no productive struggles were observed when students attempted to prove that $\frac{1}{tan^2x} + \frac{1}{tanx} = \frac{1+tanx}{tan^2x}$ and that $\frac{1}{sinx} - \frac{1}{tanx} = \frac{1-cosx}{sinx}$. Consequently these activity questions were omitted from Table 5.3. Also, no 'misconception and errors' and 'uncertainty in explaining and sense making' productive struggles were observed for DCAT 2 Activity 1 questions and they were thus omitted from Table 5.3.

Some students had issues with the FOIL method, which had been explained in both the ninth- and the tenth-grade years at their school. For both factorisation and the use of FOIL, the participating teacher lowered the cognitive demand on the students by telling them how to employ these methods.

5.2.2.4 Distribution of the students' productive struggles in DCAT 2 Activity 2 questions

Next, questions from DCAT 2 Activity 2 are analysed. Table 5.4 below summarises the distribution of the productive struggles encountered in Activity 2 DCAT 2-level questions.

Table 5.4 Distribution of students' productive struggles from DCAT 2, Activity 2 questions

Questions	**Types of productive struggle**		
Prove the following trigonometric identities	**Getting started**	**Carrying out a process**	**Total**
$\frac{1}{sinx - cosx} + \frac{1}{sinx + cosx} = \frac{2sinx}{(sinx - cosx)(sinx + cosx)}$	0	1	**1**
$(1 + \frac{1}{tan^2x})(cos^2x) = \frac{1}{tan^2x}$	1	1	**2**
Total	**1**	**2**	**3**

For DCAT 2 Activity 2 questions, no productive struggles of the form ‘uncertainty in explaining and sense-making’ and ‘misconceptions and errors ‘were observed, and they were thus omitted from Table 5.4.

5.2.2.5 An analysis of the question with the highest number of productive struggles from DCAT 2 Activity 2

Similarly to the process for DCAT 2 Activity 1, the question for DCAT 2 Activity 2 with the highest number of productive struggles is now analysed.

Only two examples of productive struggles were observed during Activity 2 questions at the DCAT 2 level. Nevertheless, most productive struggles were observed when students were asked to prove $(1+\frac{1}{tan^2x})(cos^2x)=\frac{1}{tan^2x}$. For this question, the students struggled equally with ‘getting started’ and with ‘carrying out a process’. Rohimah and Prabawanto (2019) suggested that students find proving trigonometric identities one of the most difficult topics in trigonometry, since it requires a deep understanding and application of trigonometric concepts.

The second-highest number of productive struggles was observed when the students were asked to prove that $\frac{1}{sinx-cosx}+\frac{1}{sinx+cosx}=\frac{2sinx}{(sinx-cosx)(sinx+cosx)}$. For this question, ‘carrying out a process’ was the only productive struggle observed. The students struggled with adding and subtracting algebraic fractions. This observation aligns with Victoria, Fauzi and Ananda’s (2017) view that fractions are a fundamental topic in mathematics, despite students having difficulty in learning this concept and teachers in teaching it. Jung-A, Jae-Geun and Kyeong Hwa (2013) agreed with the view that students struggle with fractions; moreover, they contended that this difficulty with fractions could be one of the reasons that students have difficulty with trigonometry.

No productive struggles were observed when the students attempted to prove that $\frac{7}{2sinx}+\frac{2}{7sinx}=\frac{53}{14sinx}$, $\frac{sinx}{cosx}+\frac{cosx}{sinx}=\frac{1}{cosxsinx}$ and $\frac{15}{3sinxcosx}+\frac{2}{5cosx}=\frac{25+2sinx}{5sinxcosx}$. Consequently, these activity questions were omitted from Table 5.4. Because so few productive struggles were observed in these activity questions, this study discusses both instances of examples of productive struggles; that is, when students were asked to prove that $(1+\frac{1}{tan^2x})(cos^2x)=$

$\frac{1}{tan^2x}$, and when the students were asked to prove that $\frac{1}{sinx-cosx}+\frac{1}{sinx+cosx}=\frac{2sinx}{(sinx-cosx)(sinx+cosx)}$.

Getting started (Prove that $\frac{1}{sinx-cosx}+\frac{1}{sinx+cosx}=\frac{2sinx}{(sinx-cosx)(sinx+cosx)}$)

S: *[writing]* $\frac{sinx+cosx}{(sinx-cosx)}+\frac{sinx-cosx}{(sinx+cosx)}$.

T: Guys? Help her out.

In this excerpt, the struggling student had reached an impasse. The teacher resolved this struggle by asking the class to intervene (successfully). The observation by the teacher is based on *student mathematical thinking.* The type of question where the teacher requests help from the class is called a *generating discussion* question. This type of questioning is important to generate classroom discussion, during which it will be easier for the teacher to notice any misconceptions that the class might have.

Getting started (Prove that $(1+\frac{1}{tan^2x})(cos^2x)=\frac{1}{tan^2x}$)

T: I'm assuming we're struggling a bit Ie...

S: A lot! *[To fellow classmates]* What was your first step? What was your idea?

In this excerpt, the teacher notices that the students are struggling, but chooses not to intervene at this stage. Roble (2017) contended that one of the consequences of productive struggles is creativity. However, this creativity comes from experimentation. The struggling student asked for help from fellow classmates to get started. The 'getting started' struggle was resolved when a fellow classmate suggested that the student replace tan^2x with $\frac{sin^2x}{cos^2x}$.

Carrying out a process

S: *[writing]* $cos^2x+\frac{cos^2x}{1}\frac{cos^2x}{sin^2x}$

C: Ah… That's not going to work.

The struggling student does not know the process for adding algebraic fractions. In this instance the struggle was resolved by both the teacher and the class; although the class chose a more cumbersome way to prove the identity, by suggesting $\frac{tan^2x}{tan^2x}\frac{cos^2x}{1}+\frac{1cos^2x}{tan^2x}$.
The teacher chose not to be the centre of learning during these activities, instead encouraging the class to participate in order to make the learning more collaborative. In this regard, Lim, Lee, Tyson, Kim and Kim's (2020) 'follow-up action' by the teacher is that of redirecting the question to the class.

5.2.2.6 Categorisation of the errors using NEA at DCAT 2 level

This section uses NEA to investigate the types of errors made by the students in the second assessment, which was based on DCAT 2 questions. The purpose of applying the NEA to the students' assessments in the activities is to establish whether there is a link between the errors that the students made when they proved trigonometric identities in the assessments, and the errors they made during their productive struggles in real time in the activities.

In Table 5.5 below, the types of errors made by the students on the second assessment are summarised. Similarly to the previous analyses, the numbers in each cell in Table 5.5 below indicate the number of times each type of error was encountered when Assessment 2 was analysed.

Table 5.5: Distribution of NEA types among assessment 2 questions at DCAT 2 level

Assessment question	Type of NEA			
Prove the following trigonometric identities	**Comprehension**	**Transformation**	**Process skills**	**Total**
$\frac{1}{cosx} - \frac{tanx}{1} = \frac{1 - sinx}{cosx}$	1	0	2	**3**
$cos^3x - sin^3x = (cosx - sinx)(1 + cosxsinx)$	3	0	7	**10**
$tan^2x - \frac{1}{cos^2x} = -1$	0	0	1	**1**
$tanx + \frac{1}{tanx} = \frac{1}{sinxcosx}$	2	0	4	**6**
$\frac{5}{sinx} - \frac{5}{cosx} = \frac{5(cosx - sinx)}{sinxcosx}$	0	1	0	**1**
$\frac{2}{tanx - 1} - \frac{2}{tanx + 1} = \frac{4}{tan^2x - 1}$	3	2	3	**8**
Total	**9**	**3**	**17**	**29**

For the second assessment, no NEA-type errors at the '*encoding'* and *'other'* hierarchies were observed. Consequently, these hierarchies were omitted from Table 5.5.
Errors of the *process skills hierarchy* occurred the most. The students made the most errors when they were required to prove that $cos^3x - sin^3x = (cosx - sinx)(1 + cosxsinx)$. For example, in Figure 5.7 below the student does not understand the process of factoring the difference between two cubes.

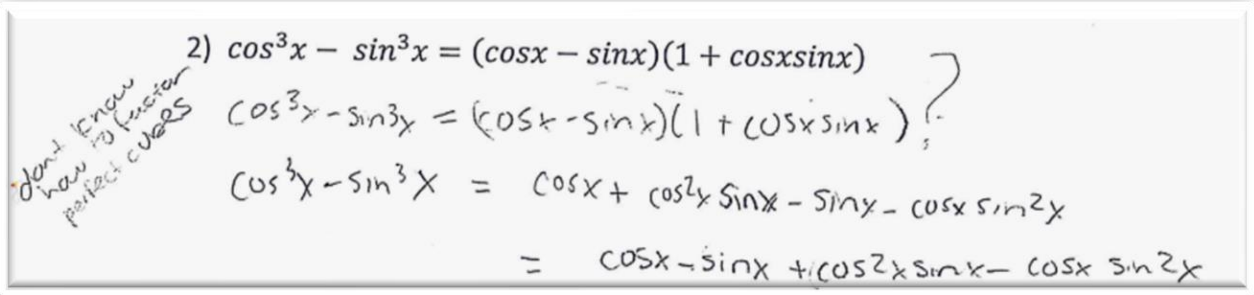

Figure 5.7: Process skills error

Burhanzade and Aygör's (2015) investigation into the difficulties that students encounter during factorisation reveals that because the students lacked mathematical fundamentals, they could not identify when to use factorisation appropriately. Burhanzade and Aygör (2015) also showed that in most cases, students struggled with factorisation that involved a trigonometric exercise. This study makes a similar observation regarding proving trigonometric identities in the second assessment.

The second-greatest number of errors occurred when students were required to prove that $\frac{2}{tanx-1} - \frac{2}{tanx+1} = \frac{4}{tan^2x-1}$. For this question, an equal number of errors occurred between the *comprehension hierarchy* and the *process skills hierarchy.* Figure 5.8 shows an error regarding the *comprehension hierarchy*, since the student did not comprehend that the question required him/her to show that the two expressions are equivalent.

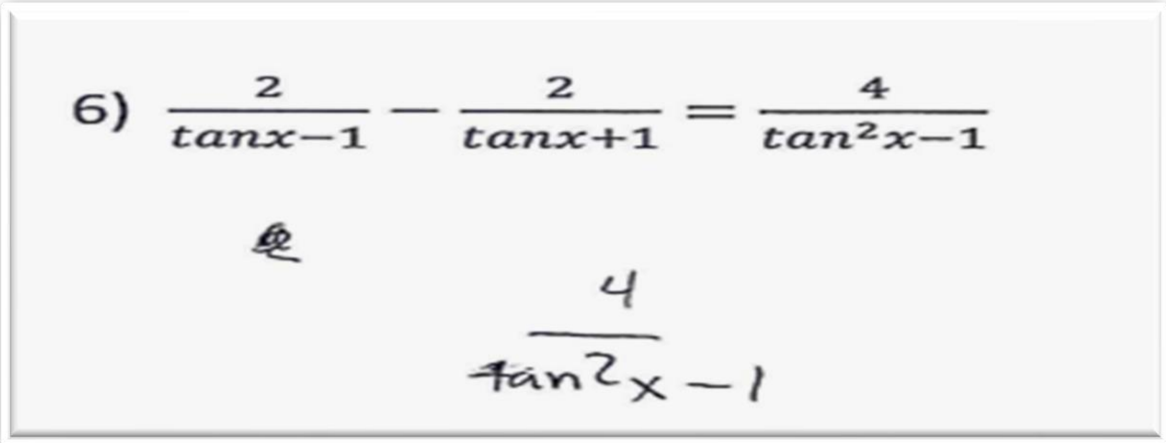

Figure 5.8: Comprehension hierarchy error

Victoria, Fauzi and Ananda (2017) claimed that irrespective of which teaching strategy is employed, students still struggle with fractions. And in her investigation into students learning trigonometric identities, Khuzwayo (2019) observed that for students, there should be a continuous review process for how to add and subtract trigonometric fractions.

Most errors made in Assessment 2 had to do with the *process skills hierarchy,* accounting for 58.6% of the total number of errors observed. Ten of the 29 errors occurred when students were required to prove that $cos^3x - sin^3x = (cosx - sinx)(1 + cosxsinx)$. In the example in Figure 5.9 below, the student chose to start the proof on the right-hand side; but then inexplicably multiplied $(cosx - sinx)(1 + cosxsinx)$. with *(cosx + sinx),* instead of using the distributive law to expand $(cosx - sinx)(1 + cosxsinx)$. This would suggest that the process of multiplying two binomials is not well established in the student's mind.
None of the productive struggles seen during the activities at the DCAT 2 level mimic this type of error.

$$(cosx - sinx)(1 + cosxsinx)(cosx + sinx)$$
$$(cosx - sinx)(cos^2x - sin^2x)(cos^2x + sin^2x + cosxsinx)$$
$$(cosx - sinx)(cos^2x + sin^2x + sinxcosx)$$
$$(cosx - sinx)(1 + cosxsinx)$$

Figure 5.9: Process skills error

The second-highest number of errors made in Assessment 2 had to do with the *comprehension hierarchy.* Figure 5.10 below shows an example of an error at the *comprehension hierarchy* since the student did not comprehend that they had to show that the two trigonometric expressions are equivalent.

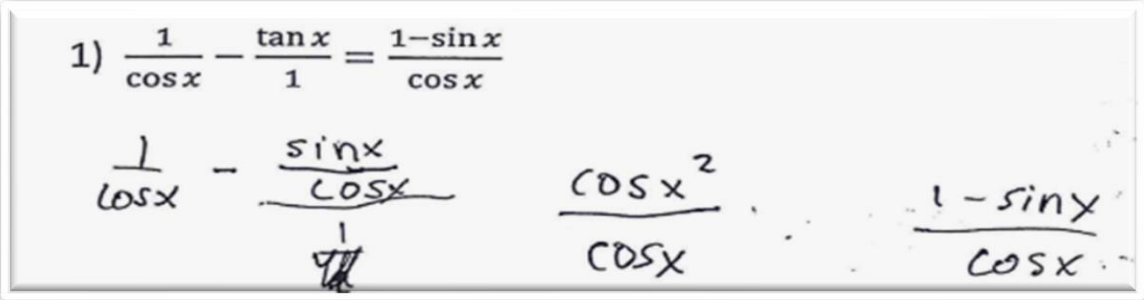

Figure 5.10: Comprehension error

The student provided no evidence for how she or he arrived at the second step. Moreover, there is no relation between the second and third step, and also the trigonometry notation is not well established.

5.2.2.7 Interviews from the focus group interview on DCAT 2 assessment and activities

This section deals with the focus group interview conducted after the second assessment at the DCAT 2 level. The main purpose of the focus group interview was to obtain evidence as to whether students' productive struggles that were observed during DCAT 2 learning activities were resolved during the assessment on DCAT 2. Focus group interviews gave the researcher a deep understanding of the nature of the errors that the students committed when proving trigonometric identities at the DCAT 2 level, and how they were related to the students' productive struggles in real time.

Moreover, from the themes that came out of the interview, the researcher could determine whether the struggles observed from the video recordings of the DCAT 2-level activities had been resolved.

The focus group met during their lunch break to discuss Assessment 2 on proving trigonometric identities. Three focus group members were present: Aud, Maud and Amy. Each member received their Assessment 2 script back to reflect on their answer choices. Assessment 2 is found in Appendix 10. Themes emanating from the focus group interviews were analysed. In the following interview excerpt, R = researcher.

Misconception of algebraic processes

Aud again demonstrated why most errors on Assessment 2 had to do with the *process skills hierarchy.*

In Figure 5.11 below, Aud did not understand the process of partial decomposition of fractions. However, she might have discovered her error if she had added the two resultant fractions. Therefore, she could have proved her assertion.

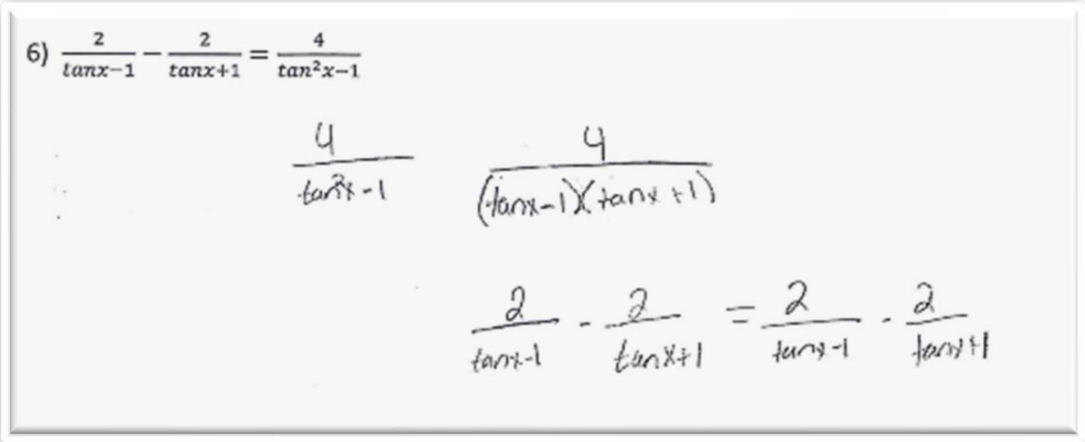

Figure 5.11: Aud's response to Question 6 on assessment 2

Maoto, Masha and Mokwana (2018) noticed the phenomenon that some students are unable to prove their assertions, and contended that reasoning and proof should be part of teaching – irrespective of the mathematical topic – to enhance students' mathematical skill if possible. Aud's reasoning can also be attributed to her lack of *procedural fluency.* The reader is reminded that according to Wilson and Heid (2011), *procedural fluency* is a student's ability to recognise how and when to apply procedures competently – in Aud's case, the ability to recognise that *partial decomposition* would be the most appropriate process for proving the trigonometric identity.

The following interview excerpt provides evidence of Aud's lack of recognition regarding the *partial decomposition* of fractions.

Aud: For number 6 *[question 6]* I just… on the right side, I just separated out the right side. Can you do that, or not?

R: Number 6? Uh, the $\frac{4}{tan^2x-1}$?

Aud: Uh-huh. *[yes]*

R: So… what do you mean, you separated it?

Aud: I did the difference between two squares at the bottom.

R: *(tanx-1)(tanx+1)?*

Aud: Yeah. And then I just separated the 4, because it could be divided by 2… *[writing]* $\frac{2}{tanx+1} + \frac{2}{tanx-1}$. Can I do that, or not?

Amy: I didn't understand the…

R: Excuse me, let me just listen to you. What do you mean, with the 4? Did you just write $\frac{4}{tanx+1} + \frac{4}{tanx-1}$?

Aud: I separated the bottom $(tanx + 1)(tanx - 1)$, and then I took the 4 and divided it by two $\frac{2}{tanx+1} - \frac{2}{tanx-1}$.

R: …minus?

Aud: *[pointing]* In the middle, right there.

R: Oh, sorry – I thought you were talking about the right-hand side.

Aud: Mmm…

R: That's left-hand side?

Aud: *[pointing]* No, this side. I changed that.

Amy: Yeah, she was changing the $\frac{4}{tan^2x-1}$. She made the right equal to the left.

R: I hear what you're saying, but… the question with number 4 is that if you do 2 times 2 equals 4, that is correct; there's nothing wrong with that. But that's multiplication. On the left-hand side you have a minus. How do you go from multiplication to minus? You can say $\left(\frac{2}{tanx+1}\right)\left(\frac{2}{tanx-1}\right)$, which is fine; but I want to know, how do you get from multiplication to minus if you move from right to left? That *is* what you're saying? Or am I misunderstanding you?

Aud: I moved from the right to the left… I don't remember how I got the minus sign.

R: Do you remember, Amy?
Amy: ...No.

The focus group can 'see' the answer

The focus group also repeated the reason they found proving trigonometric identities easier than simplifying trigonometric expressions. While the students struggled with what is considered 'simplified' when they simplified trigonometric expressions, with proving trigonometric identities they knew what the answer should be.

In their investigation into algebraic thinking and attitudes towards algebra, Siew et al. (2016) reported that students found it hard to answer questions that required the application and understanding of algebraic expressions to perform difficult operations recorded in the Trends in International Mathematics and Science Study (TIMSS). Siew et al. (2016) hypothesised that this might explain why students find proving trigonometric identities to be easier – because students know what their 'answer' should be, and find that answer by any means possible. The following interview excerpt supports the focus group's assertion that proving trigonometric identities is 'easier' than simplifying trigonometric expressions.

R: How did yesterday's board work go? I must say, I'm very impressed with all of you, from when we started the whole study until recently. You've done very well.

Aud: I think we just do better when we can see the result, and we know what to work towards.

R: So, the 'proving' part you can do better than the simplifying?

Amy: Yeah.

R: And you say that's because you can see what you're working towards?

Maud: Yeah… you know when to stop, and don't do more than… what it is. You don't have to overthink, 'Oh, can I do something else,' and go past the actual answer.

The excerpt above is based on real-time activities rather than the NEA. However, the notion of 'seeing the answer' can also be applied to the NEA.

5.2.3 Data Analyses on DCAT 3 – Critical thinking level activities

This section analyses activities that were performed at DCAT 3 level. As with the activities in Chapter Four, the questions at DCAT 3 level were the most difficult of the three levels. Thus, the students' progress and performance on the DCAT 3 level activities may be compared to their progress and performance on the DCAT 1 and 2 activities.

The only productive struggle that was observed during the Activity 1 questions at the DCAT 3 level was when the students struggled to 'get started' when proving that $\frac{sin^3x+cos^3x}{sinx+cosx} = 1 - sinxcosx$.

<u>Getting started</u>

T:	What do you need help with?
S:	*[Gesturing towards the question]*
T:	Really? Well, can any of your classmates help you?
C:	Okay, you take the top part.
S:	$\frac{(sinx+cosx)(sin^2x-sinxcosx+cos^2x)}{sinx+cosx} = sin^2x - sinxcosx + cos^2x = 1 - sinxcosx$

The teacher resolved this struggle by asking the class to help the struggling student. In this instance the teacher's intervention is based on *student mathematical thinking*.

The reader is reminded (from Chapter Two) that 'teacher noticing' as expanded by Teuscher et al. (2017) is made up of three components, namely 'attending', 'interpreting' and 'responding'. The teacher used a *generating discussion* question to assist the struggling student. In his investigation into student enrolment in mathematics classes with frequent discussion, Kosko (2012) contended that mathematics discussion has a positive impact on mathematics achievement.

5.2.3.1 Distribution of the students' productive struggles in DCAT 3 Activity 2 questions

This section analyses Activity 2 questions at the DCAT 3 level. Like the analysis of DCAT 3 Activity 1 questions, DCAT 3 Activity 2's analysis starts with the distribution of the students' productive struggles.

Table 5.6 below shows the distribution of the productive struggles the students experienced during Activity 2 questions at the DCAT 3 level. For this activity the students only struggled with two questions, as indicated in Table 5.6.

Table 5.6: Distribution of students' productive struggles from DCAT 3, Activity 2 questions

Questions	Types of productive struggle		
Prove the following trigonometric identities	**Getting started**	**Carrying out a process**	**Total**
$tanx - \frac{1}{tanx} = \frac{1-2cos^2x}{sinxcosx}$	1*	1	**2**
$\frac{1}{1+sinx} + \frac{1}{1-sinx} = \frac{2}{cos^2x}$	0	1	**1**
Total	**1**	**2**	**3**

For DCAT 3 Activity 2 questions, no productive struggles regarding ‘uncertainty in explaining and sense making’ and ‘misconceptions and errors’ were observed, and these categories were thus omitted from Table 5.6. However, the productive struggles ‘getting started’ and ‘carrying out a process’ were observed.

No productive struggles were observed when the students attempted to prove that $\frac{sin^2x+3sinx+2}{3sin^2x+7sinx+2} = \frac{sinx+1}{3sinx+1}$ and $\frac{1}{cosx+1} + \frac{1}{cosx-1} = -2\frac{cosx}{sin^2x}$. Consequently, these activity questions were omitted from Table 5.6.

When the students were required to prove that $tanx - \frac{1}{tanx} = \frac{1-2cos^2x}{sinxcosx}$ they exhibited one productive struggle more than when they were required to prove that $\frac{1}{1+sinx} + \frac{1}{1-sinx} = \frac{2}{cos^2x}$. When the students proved that $tanx - \frac{1}{tanx} = \frac{1-2cos^2x}{sinxcosx}$, the same number of productive struggles occurred for ‘getting started’ and ‘carrying out a process’. On the other hand, when the students were asked to prove that $\frac{1}{1+sinx} + \frac{1}{1-sinx} = \frac{2}{cos^2x}$, the productive struggle ‘carrying out a process’ occurred only once.

Since the students exhibited the same number of productive struggles when they attempted to prove both that $tanx - \frac{1}{tanx} = \frac{1-2cos^2x}{sinxcosx}$ and that $\frac{1}{1+sinx} + \frac{1}{1-sinx} = \frac{2}{cos^2x}$, both questions will be analysed. First, the productive struggles regarding the question: Prove $tanx - \frac{1}{tanx} = \frac{1-2cos^2x}{sinxcosx}$

<u>Getting started</u>

S: So, what am I supposed to do? Because if I multiply with anything, it’s going to change the bottom; and I can’t add this *[*$sin^2x - cos^2x$*]* together.

T: No, don’t do that.

In this excerpt, the struggling student reached an impasse. The teacher lowered the cognitive demand on the student by explaining to him what not to do, by exclaiming: “No, don’t do that.” No questions were posed by the teacher, but his response to the student’s struggle is based on the student’s *mathematical thinking.* In this instance the student shows a lack of *conceptual understanding,* exclaiming: “So, what am I supposed to do?” Although the *mathematical content* was at the DCAT 3 level, the students seemed to struggle in choosing which side of the equation to start the proof.

<u>Carrying out a process</u>

S: Oh! *[writing]* $\frac{sinxsinx-cosx}{sinxcosx}$

T: …No. Not at all. In the answer, do you have a sin^2x?

C: Is that $1\text{-}cos^2x$?

T: Hey! Hey! *[calling the class to pay attention]*

C: Good job! sin^2x can turn into $1\text{-}cos^2x$.

S: *[writing]* $\frac{1-cos^2x-cos^2x}{sinxcosx}=\frac{1-2cos^2x}{sinxcosx}$

The struggling student wrote $\frac{sinx}{cosx}-\frac{1}{\frac{sinx}{cosx}}=\frac{sinx*sinx}{cosx}-\frac{cosx*cosx}{sinx}=\frac{sin^2x-cos^2x}{sinxcosx}=\frac{sinxsinx-cosx}{sinxcosx}$, inexplicably breaking down sin^2x as *sinxsinx.* However, the class resolved this struggle by guiding the student to the correct method. The role of the teacher in this excerpt was to call the class to pay attention. Under teacher noticing he made a *general observation* that the class was not paying attention to the struggling student at the board. In this instance, the class resolved the student struggle through *probing guidance;* that is, the class used student thinking to guide the struggling student to a solution.

Next, the productive struggles regarding the question: Prove that $\frac{1}{1+sinx}+\frac{1}{1-sinx}=\frac{2}{cos^2x}$:

<u>Carrying out a process</u>

S: *[writing]* $\frac{2}{cos^2x}=\frac{2}{1-sin^2x}=\frac{2}{(1-sinx)(1+sinx)}=\frac{1}{1-sinx}+\frac{1}{1+sinx}$

T: I don't think that's absolutely right – you can't do that.

C: $\frac{1(1-sinx)}{1+sinx(1-sinx)}+\frac{1(1+sinx)}{1-sinx(1+sinx)}=\frac{1-sinx+1+sinx}{1-sin^2x}=\frac{2}{cos^2x}$

In this excerpt, the student incorrectly tries to decompose $\frac{2}{1-sin^2x}$. This shows a lack of *procedural fluency* on the struggling student's behalf. This productive struggle was resolved by the class recommending that the struggling student start the question from the left-hand side. Again, the class assisted the struggling students with difficult *mathematical content*. As with ‘carrying out a process’ in the previous question, the class resolved the student struggle through *probing guidance.* The teacher made a *student mathematical thinking* observation but did not participate in resolving the struggle. Osada and Supatmono (2020) noted that to prove trigonometric identities successfully, students can either simplify the left-hand side to look like the right-hand side, or simplify the right-hand side to look like the left-hand side.

5.2.3.2 Categorisation of errors at the DCAT 3 level using NEA

This section analyses errors that were committed during the third assessment using NEA. The assessment was at the DCAT 3 level. The purpose of applying NEA to the students’ assessments in the activities is to establish whether or not there is a link between the errors the students made when they proved trigonometric identities at the DCAT 3 level in assessment 3, and the errors they made during their productive struggles in real time in activities.

In Table 5.7 below, all the errors committed by the students in the third assessment are summarised under their specific categories. The number in each cell in the table indicates the number of times each type of error occurred.

Table 5.7: Distribution of NEA types among assessment 3 questions at DCAT 3 level

Assessment question	Type of NEA				
Prove the following trigonometric identities	Comprehension	Transformation	Process skills	Encoding	Total
$\frac{1}{1+cosx}+\frac{1}{1-cosx}=\frac{2}{sin^2x}$	1	0	0	1	**2**
$\frac{1}{cos^2x}+\frac{1}{sin^2x}=\frac{1}{cos^2x}\frac{1}{sin^2x}$	1	0	0	0	**1**
$1-\frac{sin^2x}{1+cosx}=cosx$	3	4	3	0	**10**
$\frac{sin^2x-cos^2x}{sinx-cosx}=sinx+cosx$	4	4	1	0	**9**
$\frac{sin^2x+4sinx-5}{sin^2x+sinx-2}=\frac{sinx+5}{sinx+2}$	1	0	0	0	**1**
Total	**10**	**8**	**4**	**1**	**23**

For the third assessment, no NEA type errors at the *'other'* hierarchy were observed. Subsequently, this hierarchy was omitted from Table 5.7.

Students made the most errors when they were required to prove that $1 - \frac{sin^2x}{1+cosx} = cosx$. For this question, errors concerning the *transformation hierarchy* occurred the most. In the example in Figure 5.12 below, the student chose an incorrect strategy to prove that $1 - \frac{sin^2x}{1+cosx} = cosx$. In the second step, instead of factorising the numerator to get *cosx(1 + cosx)*, the student chose to replace the number 1 in the denominator with $sin^2x + cos^2x$, and committed a misconception in terms of cancellations.

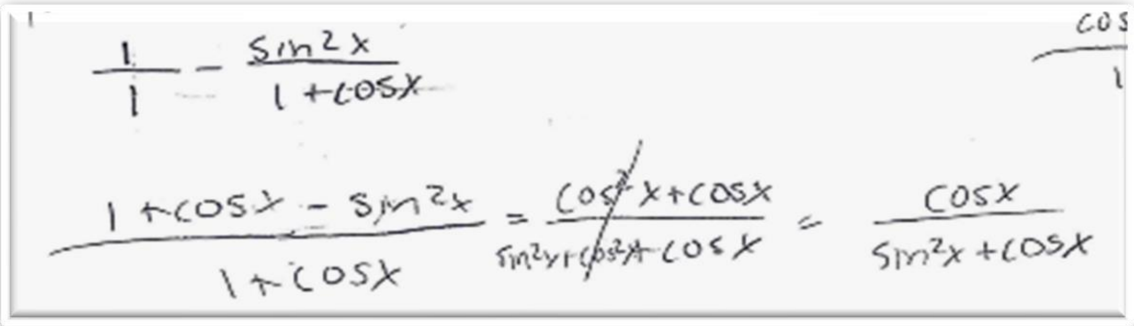

Figure 5.12: Transformation hierarchy error

Also, to prove that $1 - \frac{sin^2x}{1+cosx} = cosx$, three errors at the *comprehension* and three errors at *process skills hierarchies* occurred. In Figure 5.13 below, the student exhibits a *comprehension hierarchy* error by giving an 'answer' without explanation.

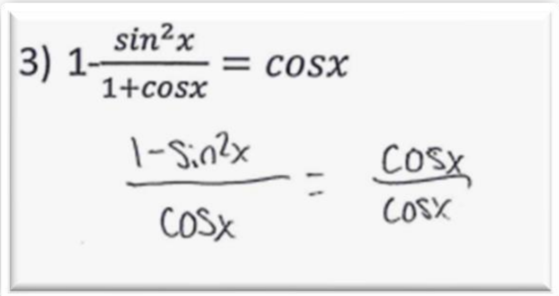

Figure 5.13: Comprehension Error

In Figure 5.14 below, the student exhibits a *process skills hierarchy* error. Although the student started off correctly by collecting a common denominator and replacing sin^2x with

$1 - cos^2x$, in the subsequent step the student inexplicably multiplied $(1 - cosx)$ by $(1 + cosx)$.

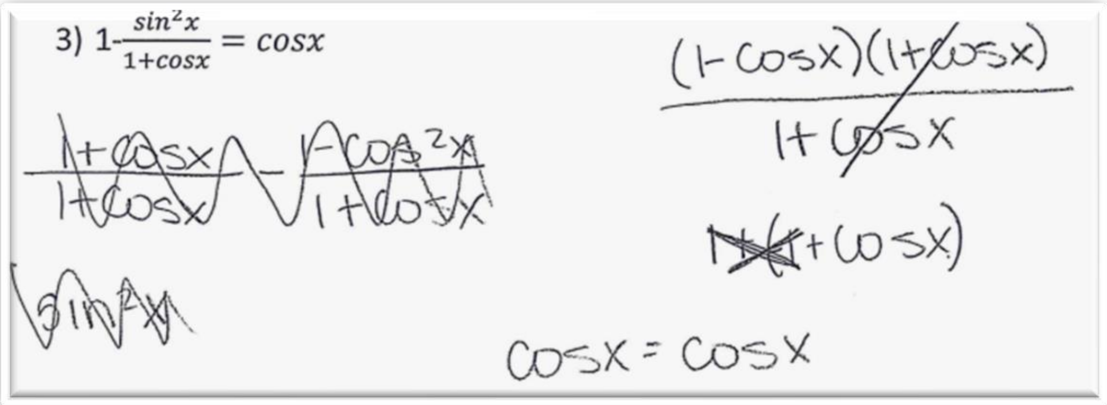

Figure 5.14: Process skills error

The second-highest number of errors occurred when the students were asked to prove that $\frac{sin^2x - cos^2x}{sinx - cosx} = sinx + cosx$. For this question, errors occurred an equal number of times regarding the *comprehension hierarchy* and the *transformation hierarchy.* In Figure 5.15 below, a student made a *comprehension hierarchy* error by incorrectly replacing $sin^2x - cos^2x$ with sin^2x and replacing *(sinx -cosx)* with *sinx.*

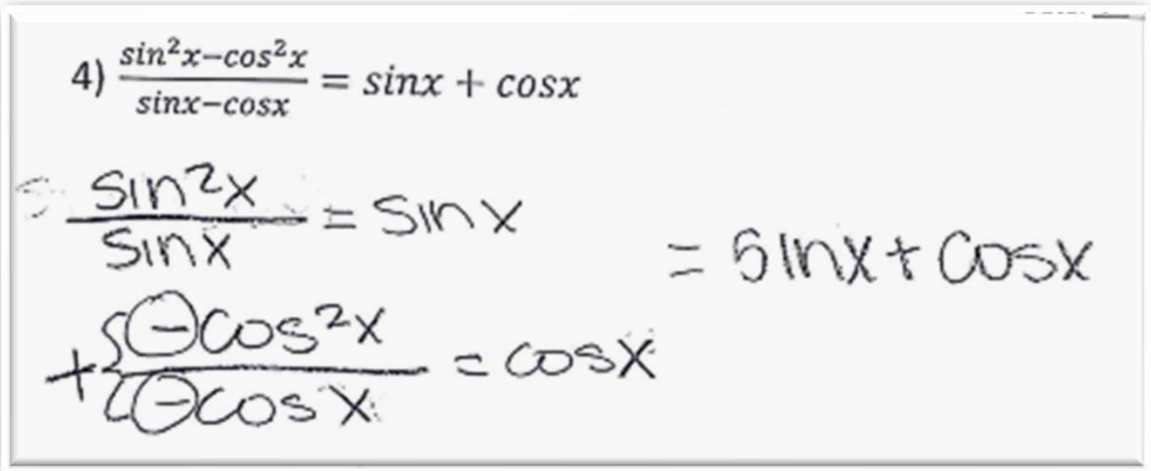

Figure 5.15: Comprehension hierarchy error

During Assessment 3, most of the errors had to do with the *comprehension hierarchy*, accounting for 43.5% of the total number of errors observed.

Four out of 10 *comprehension hierarchy* errors occurred when the students were required to prove that $\frac{sin^2x - cos^2x}{sinx - cosx} = sinx + cosx$. In the example in Figure 5.16 below, the student made a *comprehension hierarchy* error, since the error was made immediately after the student rewrote the question. The student erroneously replaced $sin^2x - cos^2x$ with -1. No similar errors were found during the DCAT 3 activities.

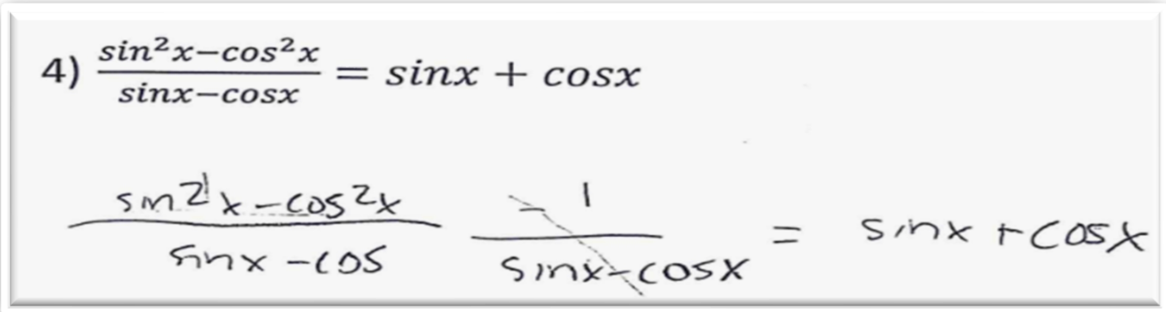

Figure 5.16: Comprehension hierarchy error

Four out of the eight *transformation hierarchy* errors occurred when the students were required to prove that $1 - \frac{sin^2x}{1+cosx} = cosx$. In Figure 5.17 below, the student employs the wrong strategy by replacing 1 with $sin^2x + cos^2x$.

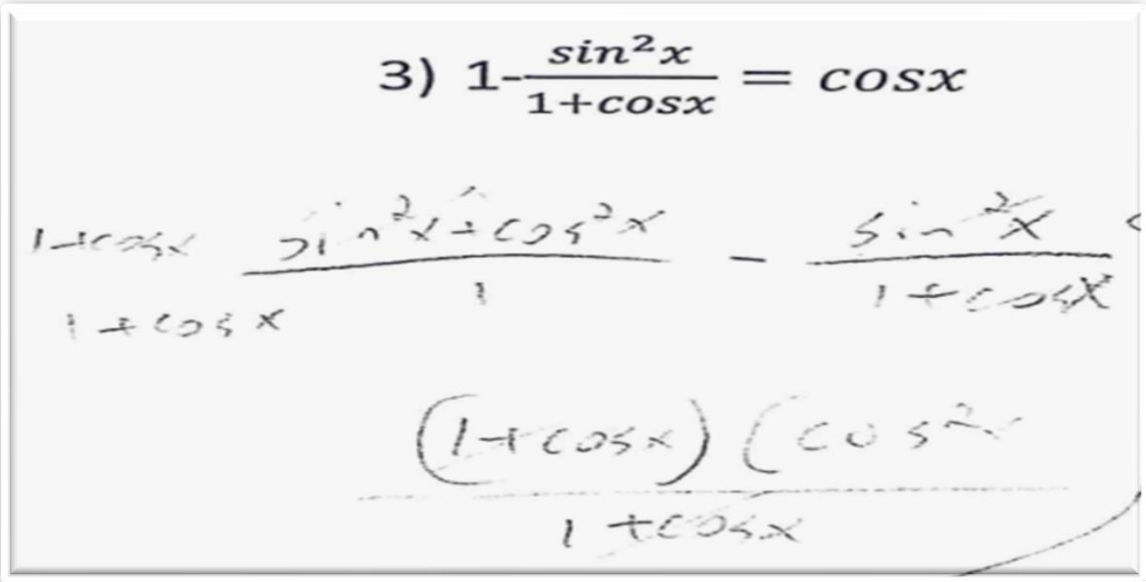

Figure 5.17: Transformation hierarchy error

For Assessment 3, most errors had to do with the *comprehension* and *transformation hierarchies.* In his study of students' mistakes when learning trigonometry, Mensah (2017) found that most errors that occurred had to do with the *encoding hierarchy*, followed by those at the *transformation hierarchy*. However, Mensah (2017) focused on students solving trigonometric ratios, first using formulae and then using right-angled triangles. In this study, Assessment 3 focused on proving trigonometric identities. Nevertheless, *transformation hierarchy* errors dominate in both studies. This would suggest that students have difficulty with 'how do I prove' questions when it comes to trigonometry.

5.2.3.3 Findings from the focus group interview on DCAT 3 activities and assessment

This section deals with the focus group interviews conducted after the third assessment, which was at the DCAT 3 level. The main purpose of the focus group interview is to get evidence on whether students' productive struggles – that were observed during DCAT 3 learning activities, were resolved during the assessment for the DCAT 3 activity questions. Focus group interviews gave the researcher a deep understanding of the nature of errors that the students committed on proving trigonometric identities at the DCAT 3 level. Moreover, from the themes that came out of the interview, the researcher could determine whether the students' productive struggles observed in the video recordings of the DCAT 3-level activities had been resolved.

Three members were present: Maud, Amy and Pam. All had their Assessment 3 scripts returned to them so they could reflect on their answer choices.
Assessment 3 can be found in Appendix 11. Themes emanating from the focus group interviews were analysed. In the following interview excerpt, R = researcher.

Algebraic manipulations

Two of the three focus group members struggled with proving the trigonometric identity $1 - \frac{sin^2x}{1+cosx} = cosx$, for different reasons. During the third assessment, most errors made in proving of trigonometric identity $1 - \frac{sin^2x}{1+cosx} = cosx$ had to do with the *comprehension, transformation* and *process skills hierarchies.* The issue of replacing 1 with $sin^2x + cos^2x$ was still problematic for some members of the focus group; they still did not comprehend which substitution for 1 to use, and under what conditions. For example, in Figure 5.18

below, Amy found the appropriate common denominator to simplify $1 - \frac{sin^2x}{1+cosx}$; however, she made a mistake in the next step.

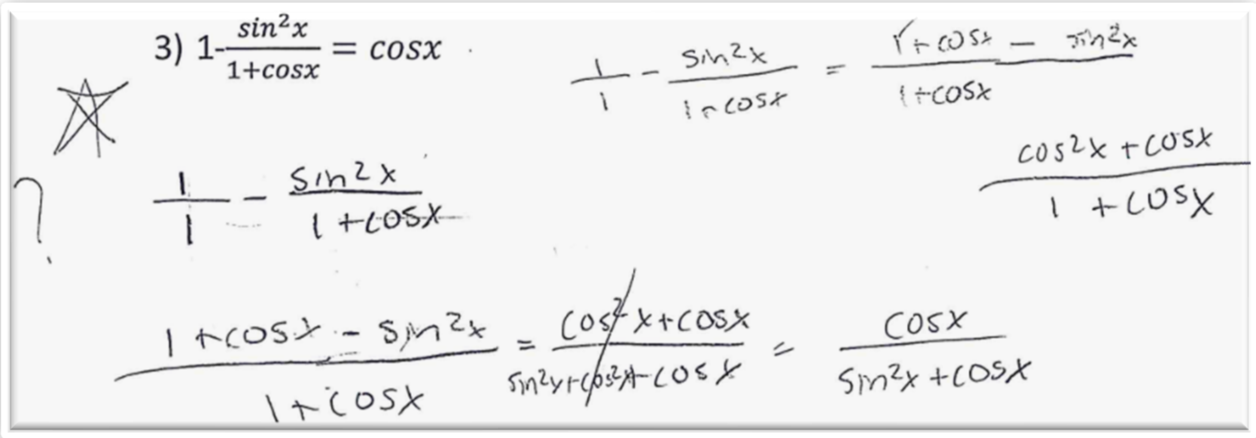

Figure 5.18: Amy's response to Question 3 on assessment 3

In her next step, instead of factorising the numerator she erroneously replaced 1 with $sin^2x + cos^2x$, which complicated the expression. The following focus group excerpt exbibits evidence of Amy's problem:

R: All of you *[struggled with]* number 3 *[Question 3]*. Let me just have a look at number 3… What was problematic about number 3?

Pam: I figured it out, how to do everything; then I got to the end – trying to make it cosine – and I had a negative in the end. I didn't know how to get rid of it. So…

Amy: I didn't know what to do – I didn't know where to start!

Pam: I didn't either.

Amy: Because I didn't know if I should do something with the 1, or…

Pam: I tried everything!

Amy: Didn't you do a question like this the other day?

Pam: …Yeah.

In Figure 5.19 below, Pam did not understand how to apply the distributive property *a(b + c) = ab + ac.*

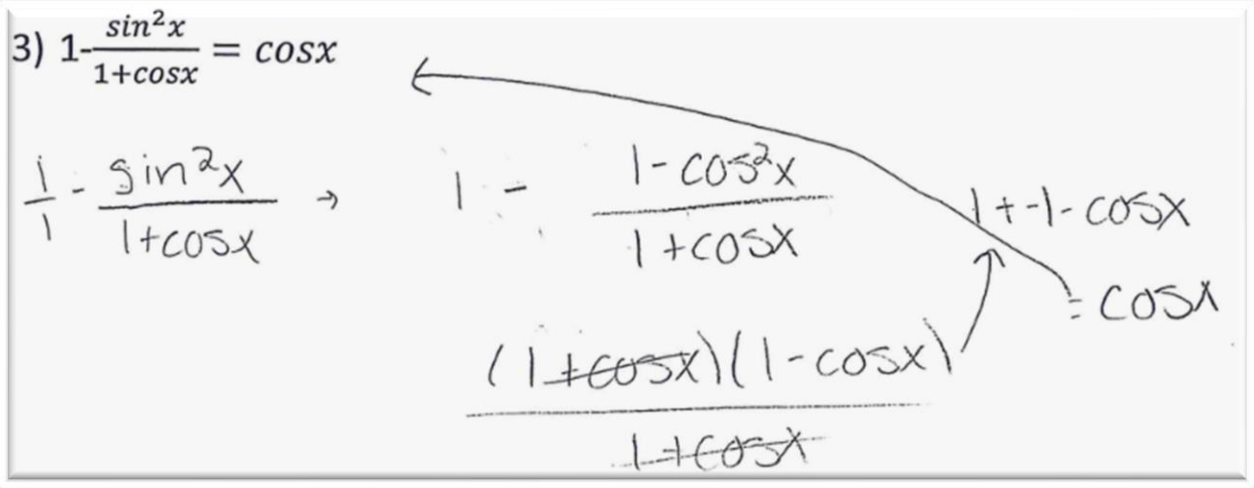

Figure 5.19: Pam’s response to Question 3 on assessment 3

Pam’s final answer should have been *1 – (1 – cosx) = 1 – 1+cosx = cosx.* However, she did not multiply the negative 1 in front of the parentheses with *-cosx.* In Chapter Two this study referred to Schoenfeld (2007), who stated that the shift from knowledge as a construct of what you know, to a construct of what you know *can apply in different domains*, is a “cognitive revolution” (p.59).

5.2.3.4 Comparison between the different DCAT-level activities when proving trigonometric identities

Next, a comparison is drawn between the types of productive struggles at the different DCAT levels.

During DCAT 1 Activity 2 questions, the students participated in a card-shuffling activity in which they were tasked with logically arranging the proofs for five questions. No productive struggles were observed during this activity.

There was an increase of eleven productive struggles from the DCAT 1 to the DCAT 2 activities. However, the questions that the students were asked to solve also became more difficult in DCAT 2. But even though the difficulty of the questions also increased from DCAT 2 to DCAT 3, the number of productive struggles decreased by seven between these two DCAT levels. In the activities in which the students proved trigonometric identities, five instances were documented of ‘getting started’, seven instances of ‘carrying out a process’, two instances where the students had ‘uncertainty in explaining and sense making’ and three instances where the students ‘expressed misconceptions’. This would suggest that ‘carrying out a process’ was most problematic for the students when they proved trigonometric identities.

5.2.3.5 Comparison between the number of types of errors for different assessments

This section tallies and summarises the errors across the three DCAT-level assessments. Table 5.8 below summarises the number of errors observed for the different assessments.

Table 5.8: Comparison between number of errors for different assessments

Assessment questions	**Type of Newman Error**				
Proving of trigonometric identities	Comprehension	Transformation	Process skills	Encoding	Total
Assessment 1	**12**	**4**	**11**	**5**	**32**
Assessment 2	**9**	**3**	**17**	**0**	**29**
Assessment 3	**10**	**8**	**4**	**1**	**23**
Total	**31**	**15**	**32**	**6**	**84**

Reading as a Newman Error type was excluded, since no examples were observed. Also, no ‘*other’* Newman Error type was observed; consequently, this category was not present in the table. There was a 9.38% decrease in the number of errors made by the students from Assessment 1 to Assessment 2, despite the difficulty level increasing from Assessment 1 to Assessment 2. Furthermore, there was an even bigger (20.69%) decrease in the number of errors between Assessment 2 and Assessment 3, despite Assessment 3 being the most difficult of the three assessments. Part of this decline in errors can be attributed to the fact that the students had a good idea what the outcome of each answer should be, irrespective of which side of the equation they had started the proof.

The reader is reminded that for the process skill hierarchy the students should perform the necessary process to prove trigonometric identities, and for the encoding hierarchy the students are required to represent their answers in a mathematical way.

Warshauer (2011) defined 'carrying out a process' as the ability to perform a mathematical process; in this case, to prove trigonometric identities, flawlessly. For the first assessment, all the questions were attempted; so 'getting started' was not problematic. Although students made the most errors at the comprehension hierarchy, they made one fewer error at the process skills hierarchy. This would indicate that the 'carrying out a process' productive struggle had not been resolved.

For the second assessment, most errors occurred at the process skills hierarchy, which again indicates that the 'carrying out a process' productive struggle had not been resolved. For this assessment, the next most errors occurred at the comprehension hierarchy.

For the third assessment, most errors occurred at the comprehension hierarchy, followed by the transformation hierarchy. The reader is reminded that for the *transformation hierarchy,* the students were expected to select an appropriate mathematical strategy to prove a trigonometric identity.

This would suggest that the 'misconception' productive struggle had not been resolved, since the students could not choose an appropriate method to prove a trigonometric identity.

5.2.3.6 Highlights from the post-teaching meeting between teacher and researcher regarding DCAT 1, 2 and 3 activities

Next, some highlights from the post-teaching meeting that was conducted between the researcher and the teacher are presented and analysed.

The purpose of the post-teaching meeting was to discuss the teacher's view on what transpired during the DCAT 1, 2 and 3 activities. The researcher and the teacher held their post-teaching meeting at the end of the DCAT level 3 activities.

The researcher and the teacher met briefly after school in the afternoon. All post-teaching meetings were transcribed verbatim. R = researcher and T = teacher. The following themes came out of the post-teaching meeting with the teacher.

The advantages of struggling.

Close to the end of this study, the teacher noticed that some struggling students were improving academically. The findings from proving trigonometric identities concur with his observation. There was a decrease in the number of student productive struggles when compared to the start of DCAT 1 Activity questions on simplifying trigonometric expressions.

The following interview excerpt shows evidence of the advantages of struggling.

T: Yeah, if we look back at the video of the kids in the eastern culture versus western culture, you know… struggling, in the western culture, is a sign of weakness; whereas it is not viewed that way in the eastern culture. Just watching these pre-calculus students and how they handle this, versus my honours geometry students… they *[the honours geometry students]* are not ready for that *[productive struggling]* at all, just because they're so used to writing notes, repeating, in a procedural *[manner]*. It's neat to see growth in students like Brigitte.

R: I told her *[Brigitte]* this morning.

T: What she's done *[showing progress academically]* is phenomenal.

R: I told her that explicitly; I'm very impressed. She showed me, when we went through some of the assessments, that she had the most difficult questions right *[correct]*.

T: That's been super-encouraging!

Most students in this study struggled with factoring. However, in Figure 5.20 below, Brigitte showed great improvement by factoring Question 5 correctly.

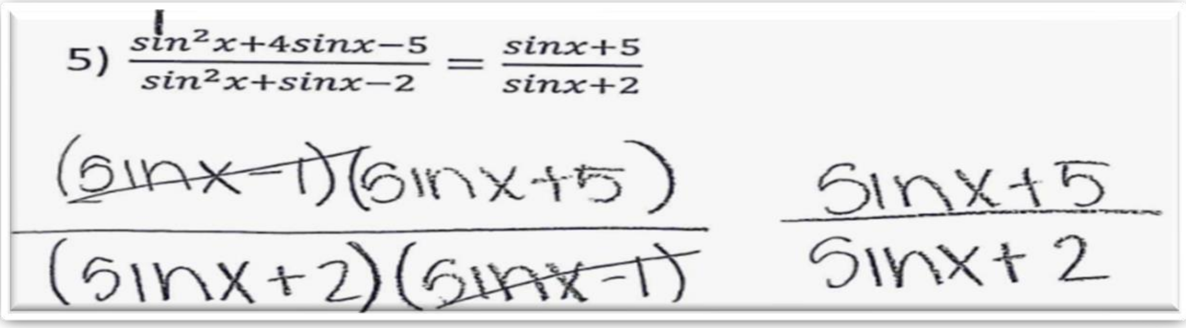

Figure 5.20: Brigitte's response to Question 5 on Assessment 3

Roble (2017) stated that productive struggle can lead to creativity. However, to accomplish this creativity takes time and experimentation. In Brigitte's case, her struggle with factoring was only resolved during Assessment 3, on the proving of trigonometric identities. However, her resilience in struggling paid off.

The influence of *didactic transposition.*

During the post-teaching meeting, the teacher noted "…that math education is very circular…". This repetition of mathematical topics may be to reinforce certain topics so that students can do well on state or national tests.

The following post-teaching meeting excerpt shows evidence of the influence of didactic transposition, where scholarly knowledge is transposed to become knowledge to be taught for a purpose; in this case, to do well on State tests.

T: I played in a golf tournament with a buddy, and talked *[for]* about 30 minutes about, that's kinda neat, what took in this study *[the teacher appreciated what this study is trying to accomplish]*. I tried to describe to him that math education is very circular; like… I tutor several middle-school students, and in geometry. I don't know in algebra, but in geometry, what I teach, a lot of the stuff that we do has already been done *[in middle school]*. It's just that we go over it deeper, and it keeps repeating and repeating in a cycle, and we don't really move that far because we're covering the same topics. But if we would expand, and… I guess what I'm saying by that is that, like, if we look at the curriculum for high-school geometry, it's very similar in what

I've seen with middle-school geometry; they cover a lot in what we have done.

R: They do?

T: Yeah!

R: I did not know that.

T: Now, they don't do it in [the] depth that we *[high-school level]* do.

In Chapter Three, Sallee and Flood (2012) pointed to the fact that policymakers in the United States prefer quantitative research where state and federal policymakers measure teacher and student performance based on statistics. It is thus of interest to teachers for their students to do well on state tests. And it is thus important that scholarly knowledge is transposed to knowledge that should be taught to align with what is required to do well on State tests. When the teacher refers to 'circular', he means that certain topics in mathematics are repeated from middle school to high school so that they can be reinforced for State tests.

5.3 Summary

In Chapter Five, the students in the study struggled less. This can be attributed – albeit not exclusively – to the following. First, the students had been exposed to trigonometric expressions (in Chapter Four) and might have learned some techniques in simplifying trigonometric expressions. Second, the students in Chapter Five could 'see' the answer to each question. This means that unlike in Chapter Four, where some students still struggled with the concept of 'simplification', in Chapter Five, whether a trigonometric identity was proven either from the right or the left-hand side of the equation, the students knew what the 'answer' had to be.

Interestingly, when all the steps to an activity question were given in scrambled form, the students did not struggle in to unscramble the solution. However, some students still struggled to write out their own solutions to some questions.

Equally importantly, the study notes that students still struggled with some basic algebraic manipulations, such as cancellation of expressions and replacing 1 with $sin^2x + cos^2x$.

In Chapter Six, this study concludes, by answering the research questions posed in Chapter One, and presenting an overview of the implications of this study.

CHAPTER SIX: CONCLUSION AND RECOMMENDATIONS

6.1 Introduction

The primary research question for this study is the following: What is the nature of the productive struggles experienced by high-school students during the *simplification of trigonometric expressions* and *proving of trigonometric identities*, and how have these productive struggles influenced the learning and teaching of trigonometry?

The primary research question is divided into three secondary research questions, referred to as research questions one, two and three. The first research question is: What types of productive struggles are observed during the *simplification of trigonometric expressions* and the *proving of trigonometric identities*? These struggles were observed during real time using pre-determined activity questions that were sequenced using the DCAT levels (Beggs & Mouw, 1980).

The second research question is: What types of productive struggles from the learning activities remained unresolved during the assessments on *simplification of trigonometric expressions* and *proving trigonometric identities*? This was achieved through the NEA and focus group interviews.

The third research question is: What are the qualitative differences in students' performance between the *simplification of trigonometric expressions* and the *proving of trigonometric identities*?

In the sections that pertain to the research questions, the data collection and analysis approaches for each research question are revisited. Since two main types of activities took place – that is, activities on the simplification of trigonometric expressions, and activities on the proving of trigonometric identities – research question one and two's major findings are presented separately, as the simplification of trigonometric expressions and the proving of trigonometric identities. For research question three, the performance of the students on the activities of simplification of trigonometric expressions and proving of trigonometric identities are compared, and the performance of the students on the assessments of the simplification of trigonometric expressions and the proving of trigonometric identities are compared. This study defines 'performance' as an *accomplishment* during the learning activities and assessments on the simplification of trigonometric expressions and the proving of trigonometric identities.

Thus, to measure the performance of students means to determine whether the students became better, stagnated, or became worse at simplifying trigonometric expressions and proving trigonometric identities.

The next section provides a summary of the contents of each chapter in this thesis.

6.2 Overview of the research

This section aims to help the reader to recapitulate the main discussions and essence of each chapter as presented in the thesis. The chapters are presented in ascending order, starting with the first chapter.

In *Chapter One*, the reasons and motivation for this study are explained. Of the essence in Chapter One is why students' productive struggles are chosen as a research topic, why trigonometry is chosen, and how the teacher can influence productive struggles through noticing and questioning. In fact, a survey of the *Journal of Research in Mathematics Education* (JRME), ResearchGate, Oxford Academic's Teaching Mathematics and its Application and SpringerLink produced no result when searched for productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities. However, there are a few pieces of research regarding trigonometry in other journals, such as the *African Journal of Educational Studies in Mathematics and Sciences, New Horizons in Education* and *Irish Educational Studies.*

This gap in the literature on productive struggles in the simplification of trigonometric expressions and proving of trigonometric identities gives this study credence, for pursuing this topic to fill these gaps in knowledge in the literature. This study could thus be used at in-service programmes to provide teaching strategies for instructing the simplification of trigonometric expressions and the proving of trigonometric identities.

The core of this study – namely the research questions – is presented in Chapter One, as well as and how this study would contribute to the teaching and learning of the simplification of trigonometric expressions and the proving of trigonometric identities. Importantly, the reader is made aware of the restrictions that COVID -19 placed on this study, and how this study dealt with the restrictions. Chapter One also describes the role of the pilot study, and how it was used to test the research instruments at a school with similar characteristics to the school that participated in the study. The theoretical frameworks, research methodology and significance of the findings were also described in Chapter One.

Chapter Two deals with the literature review on students' productive struggles, and presented a case for the choice of the ATD, as proposed by Chevallard (1992), as the main theory – where this study is foregrounded. The ATD represents mathematics as a human activity. The key tenets of the ATD are described, as well as how they relate to the construct of students' productive struggles in trigonometry. Productive struggles were also supported by other theoretical frameworks that informed this study's data collection, analysis and interpretations. For example, the definition of mathematical proficiency of Kilpatrick et al. (2001) was used to determine the relationship between the students' productive struggles and their mathematical proficiency. Schoenfeld (2017) contended that if teaching for robust understanding is to occur, students should be knowledgeable, disciplined, independent problem solvers and thinkers; and Jacobs (2010) claimed that a teacher notices interaction between students by *attending* (listening), *interpreting* and *responding* (talking). Boaler and Brodie's (2004) nine types of teacher questions were also used to determine what question types resolved a productive struggle by the students. However, the theoretical framework most often used in this study was identified by Warshauer (2011). Warshauer (2011) identified four categories of productive struggles, namely getting started, carrying out a process, experiencing uncertainty in explaining and sense making, and expressing misconceptions and errors. In addition, this study also used Warshauer's (2014) framework to categorise the teacher's responses to a struggling student. The categories for teacher response include telling, directed guidance, probing guidance, affordance, and an unfocused or vague response by the teacher.

The NEA theoretical framework introduced by Newman (1977a) was used to analyse the assessments administered to the students after the DCAT activities.

In *Chapter Three,* the research design and methodology are explained. Although Queirós et al. (2017) contended that qualitative research methodology is intended to be used to understand the meaning of actions by students in a classroom (such as when they simplify trigonometric expressions and prove trigonometric identities), in this chapter, the choice of a qualitative approach supplemented by descriptive statistics is clarified. The qualitative research methodology was used to gather data via video recordings of the activities, and audio recordings of the focus group interviews and teacher meetings. Also, the research design flow chart is given. A justification of the research sample is stated, as well as how data was collected, analysed and interpreted. Lastly, this chapter explains how the study dealt with ethical issues regarding minors.

Chapter Four focuses on an analysis of the activities and assessments for the simplification of trigonometric expressions. The sequencing of the analysis in Chapter Four was based on DCAT-informed activities. This means that the first activities that were analysed were at the DCAT 1 level. The next activities analysed were at the DCAT 2 level, and the last batch at the DCAT 3 level. Also in Chapter Four, a comparison was drawn between the students' productive struggles at the different DCAT levels in simplification of trigonometric expressions. In addition, the three assessments were also compared in terms of NEA types for the simplification of trigonometric expressions. The purpose of applying the NEA to the students' assessments in the activities is to establish whether or not there is a link between the errors that the students made when they simplified trigonometric expressions and proved trigonometric identities in the assessments, and the errors the students made during their productive struggles in real time in the activities.

Although at the start of this research study the students found it difficult to get to grips with the idea of productive struggling, as time passed the effects of productive struggling were noted, and the process seemed to bear fruit. Although not all questions were solved immediately, at least the students were starting to argue and debate among themselves about possible (not always correct) solutions.
A major point of interest in Chapter Four was the students' inability to distinguish between 'simpler' and 'more difficult'. This inability to ascertain what was 'simpler' was a hindrance to their progress in simplifying trigonometric expressions.

Chapter Five focused on the analysis of the activities and assessments for proving of trigonometric identities. The analysis started with DCAT 1-level activities, followed by the analysis of DCAT 2-level activities, and lastly the analysis of activities at the DCAT 3 level. A comparison was drawn between the students' productive struggles at the different DCAT levels for proving trigonometric identities. In addition, the NEA types were also compared between the three assessments for proving trigonometric identities. The purpose of applying the NEA to the students' assessments in the activities is to establish whether or not there is a link between the errors that students made when they proved trigonometric identities in the

assessments, and the errors students made during their productive struggles in real time in the activities.
Analysis of the data on students' productive struggles on the proving of trigonometric identities shows that the students performed much better at the proving of trigonometric identities than they did with the simplification of trigonometric expressions. One of the main reasons for this improvement in performance is that the students claimed that they could 'see' the answer. This merely means that the hindrance of 'simplifying' was absent. The students knew that irrespective of which side of the equivalent sign they started with an identity, the opposite side of the equation sign would be the answer.

In *Chapter Six,* this study attempts to answer the research questions. Chapter Six started by giving an overview of the research and then revisited the research questions as a reminder to the reader. The major findings concerning each research question are given in terms of the simplification of trigonometric expressions and the proving of trigonometric identities.
This is followed by how the ATD informed the teaching and learning strategies of trigonometry at high-school level. The ATD achieves this by viewing mathematics as a human activity. The ATD mainly uses praxeologies to describe how students simplified trigonometric expressions and proved trigonometric identities. A praxeology consists of two parts, namely the practical block and the theoretical block. The practical block is concerned with the *task type* and *technique.* For this study, the task type is how to simplify trigonometric expressions and prove trigonometric identities. The *technique* informs the task type. For this study the technique is mathematical processes and strategies that either impeded or helped the students to successfully simplify trigonometric expressions and prove trigonometric identities. Under the theoretical block, the *technology* informs the technique. For this study, the technology is the rules of algebra and equivalences. Lastly, under the theoretical block, the *theory* informs the technology. For this study, the theory is how students *learn* the simplification of trigonometric expressions and the proving of trigonometric identities.

Chapter Six also deals with directions for future research, this study's limitations, and its conclusion. A direction for future research would be to expand on the current study by focusing on any of its limitations. Based on the findings made, this study expresses whether it achieved what was intended to achieve from the beginning. The findings from this study indicate to what extent the aims and objectives of this study were achieved.
The next section revisits the research questions that were posed in Chapter One.

6.3 The Research Questions Revisited

The major findings of the first two research questions are discussed in two parts. In the first part, the major findings from the simplification of trigonometric expressions are discussed. The second part deals with the findings from the proving of trigonometric identities.
The next section re-states research question one and describes and how this study addressed it.

6.3.1 Research Question One: What types of productive struggles are observed during the simplification of trigonometric expressions and proving of trigonometric identities?

This section presents the types of productive struggles that were observed during the simplification of trigonometric expressions and proving of trigonometric identities.
For ease of understanding, the first research question is divided into two parts. One part analyses the types of productive struggles that were observed during the activities for the simplification of trigonometric expression, and the other analyses the productive struggles that were observed during the activities for the proving of trigonometric identities.
Next comes a discussion of the productive struggles that were observed during the activities for the simplification of trigonometric expressions. Learning activities on the simplification of trigonometric expressions were video recorded and transcribed verbatim, then analysed using a predetermined framework. Any students' productive struggles that emerged were noted and categorised according to the Warshauer (2011) framework.
The next section presents the results obtained from the productive struggles observed during the simplification of trigonometric expressions.

6.3.1.1 Productive struggles that were observed during the simplification of trigonometric expressions

In this section this study analyses the productive struggles that were observed during the activities for the simplification of trigonometric expressions.
The video analysis of the classroom activities revealed that the "getting started" process was further impeded by the students "misconceptions" about the simplification of trigonometric

expressions. Throughout the DCAT 1-level activities the students struggled with identifying and applying appropriate techniques needed to begin the "getting started" process of simplifying trigonometric expressions. These processes included manipulating equations; for example, how to rewrite $sin^2x + cos^2x = 1$ as $sin^2x = 1- cos^2x$.

During one of the activities, the teacher simplified $tanxcosx$ as $sinx$. This replacement for $sinx$ would suffice to simplify $\frac{cosx}{tanx} sinx$, if used properly. However, the students became confused between the appropriate substitution and the fact that they had to deal with a fraction. This confusion was exacerbated by the students' over-reliance on replacing 1 with $sin^2x + cos^2x$. This may be because the students first interaction with trigonometric identities was $sin^2x + cos^2x$ = 1.

In addition, there were misconceptions displayed about "simplification", for example when some students wanted to "break up" a trigonometric expression with an equals sign. Again, this would suggest that the concept of "simplification" is not well developed.

The students struggled to square a binomial. For this struggle, the students only squared the two terms in the binomial and did not calculate the middle term.

The students also did not know under what conditions cancellation of terms can take place. Most of the students performed cancellation based on the proximity of like terms in the numerator and the denominator.

Even when the students did not struggle to get started with an activity question, this did not guarantee that they would not reach an impasse during the solution of the question. When the students did reach an impasse after having no problem getting started, this struggle can be labelled as a *delayed impasse struggle.* That means the struggle is not immediately apparent, but becomes clearer as a student progress with a question.

Again, "misconceptions" about the process of simplification of trigonometric expressions proved a major struggle. This means that the students did not understand when the solution to an activity question was "easier" or more difficult than what they started.

Misconceptions about mathematical concepts (mostly factorisation) were the most dominant productive struggle during DCAT 2 activities. It is not that the students could not factorise an algebraic expression, per se; it is more that they compartmentalised knowledge. That means it was difficult for the students to understand, for example, that a trigonometric expression could also have a common factor – and hence can be manipulated in a similar way to algebraic expressions. The students seemed to be obsessed with memory aids such as 'keep change flip" when dealing with the division of two fractions. This means that when simplifying a quotient such as $\frac{\frac{x}{y}}{\frac{a}{b}}$, the solution would be to "keep" the fraction $\frac{x}{y}$ in the numerator and 'flip" and multiply the fraction $\frac{a}{b}$ in the denominator.

Again, the students were able to regurgitate the (algebraic) process but failed to transfer this knowledge to trigonometry. This study also observed that the students perceived trigonometry and algebra to be unrelated to one another.

The least number of productive struggles were observed during DCAT 3 activities. However, similarly to the DCAT 2 activities, in the DCAT 3 activities the students continued to struggle with executing well-known mathematical processes and procedures. For example, the students failed to correctly determine a common denominator between two algebraic fractions. Also, they became confused between taking the square root of an expression and under what conditions the exponents of a term should be divided by two.

Another observation was that productive struggles do not necessarily produce correct answers; nevertheless, the students did start to produce knowledge that could be used to simplify trigonometric expressions when working in a group. In their investigation into teaching higher-level mathematics, Yadgarovna and Husenovich (2020) found that when students collaborate in a group and support each other's productive struggles, they master the content better. At first glance, the productive struggles "getting started", "misconceptions" and "carrying out a process" may seem disjointed. However, the "getting started" productive struggle is dependent on the students' "misconceptions" and inability to "carry out known mathematical processes". From a teaching perspective, it is important to incorporate any new mathematical concepts and processes in different applications in mathematics. For instance, the manipulation of algebraic expressions can be extended to include trigonometric expressions. This means that if a teacher explains the process of adding $\frac{5}{x+1} + \frac{7}{x-9}$, for example, the next example could be $\frac{5}{sinx+1} + \frac{7}{sinx-9}$.

The next section presents results obtained from the productive struggles observed during the proving of trigonometric identities.

6.3.1.2 Productive struggles that were observed during the proving of trigonometric identities

This section presents results for the productive struggles that were observed during the proving of trigonometric identities.

There was a reduction in the number of productive struggles from the simplification of trigonometric expressions to the proving of trigonometric identities. The reason for this reduction was revealed during the focus group interview. The focus group members contended that they could “see” what they had to work towards, whereas with the simplification of trigonometric expressions they did not know where to “stop”.

On the whole the students were unsure which side of the equation to start with when they had to prove a trigonometric identity. In addition, the over- (and sometimes quite unnecessary) use of the trigonometric identity $sin^2x + cos^2x = 1$ still prevailed.

The students’ struggle with adding and subtracting algebraic fractions and factorisation continued. In particular, the factorisation of the sum of or difference between two cubes proved challenging. In addition, it was apparent that the transfer from algebra to trigonometry of the concept of the difference between two squares was not well developed. From a teaching perspective, additional time should be spent to inculcate how to calculate the sum of and difference between two cubes, not only in algebra but also in other branches of mathematics such as trigonometry.

Next, research Question Two is restated, with a discussion of how this study addressed it.

6.3.2 Research Question Two: What types of productive struggles from the learning activities remained unresolved during the assessment on simplification of trigonometric expressions and proving trigonometric identities?

This section presents the types of productive struggles from the learning activities that remained unresolved during the assessments on the simplification of trigonometric expressions and the proving of trigonometric identities.

For ease of understanding, the second research question is divided into two parts. One part analyses the types of productive struggles from the learning activities that remained unresolved during the assessments on simplification of trigonometric expressions. The other part analyses the types of productive struggles from the learning activities that remained unresolved during the assessment on proving trigonometric identities**.**

The data comes from the documentary analyses of the assessments that were written when students had completed each set of DCAT activities. The student errors on the assessment were categorised (based on a pre-determined NEA framework) as errors made under the *reading, comprehension, transformation, process skill,* or *encoding hierarchy* categorisations. The purpose of applying the NEA to the students' assessments in the activities was to establish whether or not there was a link between the errors the students made when they simplified trigonometric expressions and proved trigonometric identities in the assessments, and the errors they made during their productive struggles in real time in the activities.

Next is a discussion of the types of productive struggles from the learning activities that remained unresolved in the assessment for simplification of trigonometric expressions.

6.3.2.1 Productive struggles from the learning activities that remained unresolved during the assessments for the simplification of trigonometric expressions

During the first assessment the students continued to struggle with carrying out well-known high-school mathematical processes, including cancellation of terms, manipulating algebraic fractions, and the unnecessary use of $sin^2x + cos^2x = 1$. The students also did not know how to simplify their answers correctly. This means they were unsure of what "simplified" means. Findings from the three focus group interviews showed that the students struggled with cancellation. These cancellation errors stemmed from the fact that the students replaced 1 with $sin^2x + cos^2x$ (sometimes unnecessarily), and then incorrectly cancelled one of $sinx$ or $cosx$. The focus group gave vague and unrelated reasons for their inability to cancel. For example: "… so I could just get done with it" or "I feel dumb" [for cancelling incorrectly]. This would indicate their deep-seated misconceptions about manipulating algebraic fractions. The focus group revealed that the notion of "simplification" is still not well developed. Some of the students replaced 1 with $sin^2x + cos^2x$ because they felt that the answers to the activity questions should be more difficult. Other members of the focus group stated that the overuse of $sin^2x + cos^2x$ stemmed from the fact that $sin^2x + cos^2x$ was the one most often used in class.

Again, the students *compartmentalised* their knowledge. In trigonometry it is not wrong to replace 1 with $sin^2x + cos^2x$; however, where to use the trigonometric identity to make an activity question simpler became the challenge for the students.
These errors on the first assessment hindered the students' ability to "carry out known processes" correctly during the activity questions. For example, to simplify the trigonometric expression $tan^2x + 1$, instead of writing tan^2x as $\frac{sin^2x}{cos^2x}$ and 1 as $\frac{cos^2x}{cos^2x}$ and thus finding a common denominator, most students wrote 1 as $sin^2x + cos^2x$, thus complicating the trigonometric expression instead of simplifying it.

The compartmentalisation of knowledge continued during the second assessment. However, the *identification* of quadratic expressions and the *factorisation* thereof in a new context was a struggle for the students. Consequently, this inability to factorise a trigonometric quadratic expression caused the students to present their answers incorrectly. Members of the focus group revealed that they did not consider factoring, which would suggest that they might not have identified a quadratic in a new context. Furthermore, the concept of "simplification" continued to be challenging for the students. Some of the students in the focus group contended that their solutions could not be that simple, or that the answer should be harder – despite the topic being "simplification of trigonometric expressions". This would suggest that the students still perceived trigonometry as a "foreign mathematical language" that is not related to algebraic knowledge.
Again, this inability of the students to identify a quadratic in a new context prevented them from "carrying out a factorisation process" in the learning activities.

The students' inability to add or subtract algebraic fractions, the order of operations and the students' inability to cancel terms correctly were the main concerns during the third assessment. The focus group revealed that the injudicious application of mathematical processes such as finding a common denominator was a cause of concern.
This inability of the students to perform known algebraic processes may have been a symptom of their deep-seated misconceptions regarding using prior knowledge in new contexts, which may consequently have stifled their ability to carry out these processes in a new mathematical context. Consequently, teachers should direct their focus towards teaching students to apply known processes in new mathematical contexts such as trigonometry.

Next is a discussion of the types of productive struggles from the learning activities that remained unresolved during the assessment for the proving of trigonometric identities.

6.3.2.2 Productive struggles from the learning activities that remained unresolved during the assessments for the proving of trigonometric identities

This section discusses the types of productive struggles experienced in the learning activities that remained unresolved during the assessment for proving trigonometric identities.
For the first assessment, students continued to struggle with adding two algebraic expressions. However, some students started to use the trigonometric identity $sin^2x + cos^2x = 1$ correctly and appropriately. Nonetheless, some members of the focus group were under the impression that $sinx + cosx = 1$, which they derived from (incorrectly) taking the square root of $sin^2x + cos^2x$. This type of incorrect thinking became more evident when some students wrote $\frac{4}{tan^2x-1}$ as $\frac{4}{tanx+1} - \frac{4}{tanx-1}$. This way of thinking suggests that some students are confused between factoring the difference between two squares and dissolving an expression. These errors from the assessment would suggest that some productive struggles regarding "misconception" have not been resolved.
In addition, unlike in the simplification of trigonometric expressions, for the proving of trigonometric identities the students knew what their solution should be. Yet some students still struggled with adding algebraic fractions. Again, the inability to add algebraic fractions correctly would suggest that the productive struggle "carrying out a process" was unresolved.

Factorisation, especially factoring the difference between two cubes, was the main concern during the second assessment. This is mainly due to the students either not knowing the formula for factoring the difference between two cubes, or having forgotten the formula, or not being able to identify factoring the difference between two cubes. Whatever the case, the productive struggle "carrying out a process" to factorise an expression presented as the difference between two cubes was still unresolved.

For the third assessment, the students struggled to add or subtract algebraic fractions. In this assessment one fraction was a whole number and the other was of an algebraic nature. The focus group revealed that a lack of exposure to these types of algebraic manipulations caused them to struggle. Here, the productive struggle “to carry out the process” of adding algebraic fractions was unresolved. The focus group revealed what the results from this study suggest that a lack of exposure to mathematical concepts in new “situations” such as trigonometry hampers the understanding of the proving of trigonometric identities.

Next, research Question Three is revisited and addressed.

6.3.3 Research Question Three: Is there any difference in student performance between the simplification of trigonometric expressions and the proving of trigonometric identities?

This section addresses a research question that seeks to establish whether there was a difference in student performance between the simplification of trigonometric expressions and the proving of trigonometric identities. In the context of this study, “student performance” refers to an *accomplishment* during the learning activities and assessments on the simplification of trigonometric expressions and the proving of trigonometric identities. To measure the performance of students means to determine if the students became better, stagnated, or became worse at simplifying trigonometric expressions and proving trigonometric identities.

Data was obtained from learning activities on the simplification of trigonometric expressions and the proving of trigonometric identities, which were video recorded, transcribed verbatim and analysed using a pre-determined framework for students' productive struggles. Data also came from the documentary analyses of the assessments written when students had completed each set of DCAT activities, using a pre-determined NEA framework as well as data from the focus group interviews.

The study will present a narrative using descriptive statistics on the differences in performance for the simplification of trigonometric expressions and the proving of trigonometric identities.

Next is discussed the performance of the students during activities on the simplification of trigonometric expressions and the proving of trigonometric identities.

6.3.3.1 Student performance during learning activities for the simplification of trigonometric expressions and the proving of trigonometric identities.

In this section, the students' performance during learning activities for simplifying trigonometric expressions and proving trigonometric identities is discussed. To measure the performance of students means to determine if the students became better, stagnated, or became worse at simplifying trigonometric expressions and proving trigonometric identities. In this regard, this study presents what the students did differently or the same, in terms of mathematical thinking, ways of working, interpretation of questions and procedural fluency in the learning activities.

There was an overall decrease in the number of productive struggles from the simplification of trigonometric expressions to the proving of trigonometric identities at the DCAT 1 level. For the DCAT 1 level activities on the simplification of trigonometric expressions, the students struggled to apply basic trigonometric identities to simplify a question. On the other hand, no productive struggles were observed for the proving of trigonometric identities at the DCAT 1 level.

However, at the DCAT 2 level there was a slight increase in the "getting started" and "uncertainty in explaining and sense making" productive struggles from simplifying trigonometric expressions to proving trigonometric identities. Although the students struggled to get started with factorisation of quadratics in the simplification of trigonometric expressions activities, they continued to struggle to get started with the factorisation of cubes and difference between two squares in the proving of trigonometric identities activities. In addition, for the proving of trigonometric identities, the addition of algebraic fractions was still a challenge for the students. The slight increase in uncertainty in explaining and sense making of a question from the simplification of trigonometric expressions to the proving of trigonometric identities could be attributed to the students' inability to recognise when to factorise a trigonometric expression in order to either simplify it or use factorisation to prove an identity.

There was an overall decrease in the number of productive struggles from the simplification of trigonometric expressions to the proving of trigonometric identities at the DCAT 3 level. For this level, the students continued to struggle to get started with factorisation of quadratics, taking out a common factor and addition of algebraic fractions when they simplified

trigonometric expressions. However, for DCAT 3-level activities the uncertainty in explaining and sense making was the addition of algebraic fractions.
On the other hand, at the DCAT 3 level for the proving of trigonometric identities, the students struggled to get started with factorisation of the sum of two cubes and the addition of algebraic fractions. "Uncertainty in explaining and sense making" describes the addition of algebraic fractions.

There was an overall decrease in the number of productive struggles from the simplification of trigonometric expressions to the proving of trigonometric identities. This could be attributed to the fact that the students may have learnt certain techniques and processes from the simplification of trigonometric expressions.

In the focus group interviews the students gave reasons as to why they struggled less with proving than with simplifying; one reason was that proving gave that an idea of what the answer would look like.

The performance between the simplification of trigonometric expressions and the proving of trigonometric identities in the assessments is discussed next.

6.3.3.2 Students' performance in the simplification of trigonometric expressions and proving of trigonometric identities in assessments

This section was analysed through the documentary analysis of the students' written assessments, and data from focus group interviews. To measure the performance of students means to determine if the students became better, stagnated, or became worse at simplifying trigonometric expressions and proving trigonometric identities. In this regard, this study presents what the students did differently or the same in the assessments in terms of mathematical thinking, ways of working, interpretation of questions and procedural fluency.

The reader is reminded that this study used the NEA hierarchies, namely *comprehension, transformation, process skill* and *encoding* to compare the students' performance in the simplification of trigonometric expressions and the proving of trigonometric identities.

When comparing the first assessment on the simplification of trigonometric expressions with the first assessment on the proving of trigonometric identities, apart for the *comprehension hierarchy* there was an overall reduction in errors (from 51 total errors for the simplification of trigonometric expressions to 32 total errors for the proving of trigonometric identities) at all the other Newman hierarchies. A possible explanation for the increase in comprehension errors is that the students did not get to grips with the notion of “proof”.

The biggest decrease in errors between the simplification of trigonometric expressions and the proving of trigonometric identities occurred when the second assessment for the simplification of trigonometric expressions was compared with the second assessment for the proving of trigonometric identities (from 104 total errors for the simplification of trigonometric expressions to 29 total errors for the proving of trigonometric identities). Although there has been an overall decrease in the number of errors between the simplification of trigonometric expressions and the proving of trigonometric identities for Assessment 3 questions (from 55 total errors on the simplification of trigonometric expressions to 23 total errors on the proving of trigonometric identities), there has been an increase in the number of *transformation hierarchy* errors (from 1 error at the transformation hierarchy for the simplification of trigonometric expressions to eight errors at the transformation hierarchy for the proving of trigonometric identities). For the *transformation* hierarchy, the students chose increasingly incorrect strategies to prove a trigonometric identity that required factorisation.

The overall reduction in errors could be attributed to the fact that the students knew what the answer should be when they proved trigonometric identities, whereas for the simplification of trigonometric expressions they were still uncertain about the concept of “simplification”. Also, with the simplification of trigonometric expressions the students might have developed their own knowledge (albeit not totally proficient) about certain mathematical processes.

There was an improvement in student performance seen when comparing total number of errors committed over the three assessments (from 210 total errors on the simplification of trigonometric expressions to 84 total errors on the proving of trigonometric identities). Again, this reduction might be attributed to the students becoming comfortable with using knowledge acquired from the simplification of trigonometric expressions.

However, there are still unresolved productive struggles, such as the inability to recognise how and when to factorise an expression. Nevertheless, this study concurs with Daily (2021) that the goal of productive struggles is not to always solve all problems correctly. Rather, as Vazquez et al. (2020) argued, the goal is for students to build their own knowledge, and not just produce correct answers.

Next, this study briefly describes how the ATD informed the learning and teaching of trigonometry at high-school level.

6.4 Primary research question: **What is the nature of the productive struggles experienced by high-school students during the simplification of trigonometric expressions and the proving of trigonometric identities, and how do these productive struggles** influence **the learning and teaching of trigonometry?**

New knowledge was generated on students' productive struggles with the simplification of trigonometric expressions and the proving of trigonometric identities, thus filling a gap in the literature that was identified in Chapter One. The *delayed impasse struggle* can be used both nationally and internationally to establish if this productive struggle can be identified in new studies on productive struggles, in topics other than the simplification of trigonometric expressions and the proving of trigonometric identities.
In addition, this study identified all the productive struggle types from Warshauer's (2011) productive struggle framework in simplifying trigonometric expressions and proving trigonometric identities.

Mensah (2017) contended that students do not understand the level of complexity of trigonometric questions, since teachers do not explain the simplification of concepts. Considering this contention by Mensah (2017), this study observed that students performed better at learning activities on proving trigonometric identities *after* they had *first* learned how to simplify trigonometric expressions. However, it was not clear that the students understood the proving of trigonometric identities better than they understood the simplification of trigonometric expressions, since the knowledge and skills needed to prove trigonometric identities were experienced first when the students simplified trigonometric expressions. It may or may not be the case that if this study had started with the proving of trigonometric identities, the simplification of trigonometric expressions would have proven easier. It could also be the case that if a mathematical test for "simpler" was incorporated in the simplification of trigonometric expressions, the students would have struggled less with the simplification process. However, what is evident even without the mathematical test of “simpler” is the students’ inability to manipulate algebraic processes. In particular, the factorisation of trigonometric expressions and the addition of algebraic fractions was a challenge for the students. Thus, even though the students in this study were all from the top mathematical echelon at the participating school, it is not automatic that having a good understanding of algebra will equate to achievement in trigonometry.

In this study the productive struggles during the simplification of trigonometric expressions and the proving of trigonometric identities were resolved by the teacher, the struggling student, and the class working collectively to solve any activity questions. However, the teacher mostly answered any questions from the class or struggling student with responses such as “Why?” and “What do we think?” Although these questions did not resolve any struggle immediately, they raised the cognitive demand on the struggling students as well as on the rest of the class. Still, certain struggles could not be resolved. The formulae for factorising the sum of or the difference between two cubes is something that has to be taught. No amount of questioning or noticing from the teacher would have been able to inculcate these formulae. In this regard, this study wishes to extend DeJarnette’s (2014) notion of *collaborative struggle* to encompass the teacher and the class, though it was not the intention of this study to look at the constructs of teacher questioning and noticing per se, but rather to discuss them in the context of productive struggles.

6.5 The ATD and the learning and teaching of trigonometry at high-school level

This section explains the role of the ATD proposed by Chevallard (1992) in the teaching and learning of trigonometry at high-school level. The ATD is about mathematics as a human activity that is learned in context. In this regard, for this study, the *task type* was simplifying and proving. The findings from this study have shown that the incorrect use of the *technique,* that is, expanding binomials, performing cancellations of terms and expressions, operations with algebraic fractions, factorisations, and the effective use of trigonometric identities can impede students’ ability to complete the *task type* successfully. These *techniques* used to simplify trigonometric expressions and prove trigonometric identities are informed by the *technology*, which is the rules of algebra and their equivalences. The *theory* that informs the technology is the application of the fundamental trigonometric identities $sin^2x + cos^2x = 1$ and $tanx = \frac{sinx}{cosx}$ in simplifying trigonometric expressions and proving trigonometric identities.

In the following section, the significance of this study is discussed.

6.6 Significance of this study

This section discusses the significance of this study regarding who benefits from the findings and what contribution this study makes to the learning and teaching of trigonometry – with specific emphasis on students' productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities.

Mathematics teachers, teacher educators, mathematics subject specialists and mathematics in-service training coordinators could use the findings of this study to adjust lesson plans or training programmes to incorporate measures to support students' productive struggles during the simplification of trigonometric expressions and the proving of trigonometric identities. While the findings from this study cannot be generalised, they can still be used in contexts informed by the South Carolina mathematics curriculum.

This study has also shown that students can coordinate with one another, as well as learning from the teacher, if this scenario is coordinated correctly. Considering that learning is a process of gaining new knowledge and expanding previously learnt knowledge, interaction with peers is one way not only to learn new strategies but also to support students' productive struggles to simplify trigonometric expressions and prove trigonometric identities.

The findings from this study provide insight into how productive struggles could be used as a teaching strategy to improve the understanding of how to simplify trigonometric expressions and the proving of trigonometric identities.

The use of Newman Error Analysis to analyse students' written scripts from the assessments provided the researcher with the opportunity to see what types of productive struggles from the learning activities remained unresolved during the assessments of the simplification of trigonometric expressions and the proving of trigonometric identities. It is beyond the scope of this study to reveal all the types of errors students make when simplifying trigonometric expression and proving trigonometric identities; however, this study attempted to identify the errors most often made by students.

6.7 Contribution to the body of knowledge

Currently, searching for ‘productive struggles in trigonometry yields no results in the *Journal of Research in Mathematics Education* (JRME), ResearchGate, Oxford Academic’s *Teaching Mathematics and its Application*, or SpringerLink. However, there are a few pieces of research in other journals. In the *African Journal of Educational Studies in Mathematics and Sciences,* Maharaj (2008) suggested that for better understanding of the $sine$ rule, we should verbalise the trigonometric identity $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$ (also known as the $law\ of\ sines$: “in any triangle ΔABC, the ratio formed by taking the length of a side and dividing by the $sine$ of the angle opposite this side is constant” (p.405). A search of the journal *New Horizons in Education* for productive struggles in trigonometry, volume 57, number 1 (May 2009) yielded ‘Trigonometric learning’ by Gür (2009). This article discusses types of errors and misconceptions encountered in trigonometry lessons. Furthermore, Usman and Hussaini (2017) reported that irrespective of their different cognitive abilities, students are susceptible to errors in trigonometry. *Irish Educational Studies* offered ‘What subject matter knowledge do second-level teachers need to know to teach mathematics? An exploration and case study’ by Walsh, Fitzmaurice and Donoghue (2017). This article discusses the readiness of pre-service mathematics teachers to teach trigonometry, i.e., whether they have enough subject matter knowledge. In turn, Kim (2013) studied the graduate teaching assistant’s mathematical understanding of teaching trigonometry.

While Moore (2012) studied quantitative reasoning in trigonometry, Moore (2009) investigated students’ conception of angle measure. Delice (2002) investigated the simplification of trigonometric expressions, contrasting Turkish and English learners’ understanding of the topic.

Hirsch, Weinhold and Nichols (1991) contended that trigonometry instruction does not align with the aims of the NCTM standards, and discouraged teachers from instructing trigonometry “as memorisation of isolated facts and procedures and proficiency with paper-and-pencil tests” (p.98), suggesting they move towards “programmes that emphasise conceptual understanding, multiple representations and connections, mathematical modelling, and problem-solving" (p.98).

None of these studies included the simplification of trigonometric expressions and the proving of trigonometric identities. The main (new) contribution by this study is students' productive struggles in the simplification and proving of trigonometric identities in real time. In this regard this study contributes to these academic and educational repositories to help future mathematics teachers with teaching strategies, by making trigonometry easily digestible for students, and more manageable for teachers.

Sayster and Mhakure (2020) used the Anthropological Theory of the Didactic (ATD) as a theoretical framework when they investigated productive struggles experienced when students simplified algebraic rational expressions. This study is intended to expand on using the ATD as a theoretical framework in productive struggles in mathematics education on the topic of trigonometry, with a focus on the simplification of trigonometric expression and proving trigonometric identities.

The findings of this study can be used to improve the teaching of trigonometry of pre-service (teacher education) teachers and in-service teachers, through dedicated workshops and seminars. For example:

- Part of a pre-service and in-service teachers' workshop or seminar could include topics like: "How to identify and address misconceptions regarding algebraic concepts when teaching *simplification of trigonometric expressions* and the *proving of trigonometric identities*".
- The results of this study can then form a blueprint on how initiate teacher questioning and responses during the instruction of the simplification of trigonometric expressions and the proving of trigonometric identities to extract student misconceptions about algebraic concepts.
- The student responses can also be used during these workshops and seminars to further understand why these deep-seated misconceptions and misunderstandings about algebraic concepts occur.
- By brainstorming why students struggled with the simplification of trigonometric identities and proving of trigonometric identities, pre-service and in-service teachers can initiate best practices to address these struggles in their instruction.

In the next section, the limitations of this study are presented.

6.8 Limitations

This section presents this study’s limitations. The reader is advised that this research was conducted at only one school, and on a small sample, for the following reasons.

The county where this study took place is small. It is home to only four high schools, at one of which the researcher is a staff member. A mathematics faculty member at the school where the researcher is based agreed to teach the activity questions. An invitation to participate in the study was sent to the other three schools. None of the schools responded to the researcher's request. A second request was sent to one of the neighbouring schools to ask if they would perform a pilot study. Again, there was no response. The researcher asked the participating school's headmaster to call the neighbouring school's headmaster and request permission to conduct the pilot study there. The neighbouring school's headmaster reluctantly agreed. After the pilot was completed, the researcher decided to continue with the study at his own school, while continuing to reach out to the neighbouring schools.

At the end of the 2018-2019 school year (May 2019), the study at the participating high school was completed. At the start of the 2019-2020 school year (August 2019) the researcher reached out to the neighbouring school once again to see if they would be willing to participate in the study. Again, there was no response.

On top of these setbacks, in April 2020 the onset of the COVID-19 pandemic ended in-person instruction in the district. Due to COVID-19 protocol concerns, the headmasters at the other high schools in the county again refused permission to perform this study at their schools. In conjunction with the scheduling issue, this brought this study to a halt. For the 2020-2021 school year, state-wide government decisions continued to impact this study. School districts have been prohibited from imposing a mask and vaccination mandate for either staff or students. Consequently, this has increased the rate of COVID-19-related absences, which has impeded the continuation of the study.

Regarding the sample size, the size of the school where the study took place is small (in the year of the study, the entire school population was close to 500 students). Consequently, the number of students enrolled in higher-order mathematics courses is also small. Only 15 students were enrolled in trigonometry in the year that this study took place.

These students were instructed by one teacher. It was thus not possible to compare a similar group of students taught by a different teacher, to see if productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities would yield different results in another group. In addition, the students were not used to the idea of teaching in an environment where students are allowed to "struggle" with the content. Also, the teacher who facilitated the activities on productive struggles did not have the necessary skills to do so, as he was not familiar with the construct. Moreover, both students and the facilitating teacher had no experience of the construct of productive struggles before this study.
To address the secondary research question, themes from the focus group interviews were used to corroborate some assertions about why students struggled with certain concepts. However, because of the students' conflicting schedules, some of the focus group interviews could only be held a day or two later. This could have a created a situation in which some students forgot why they had written some answers. By being so overtly candid about what was expected in the class, it is possible that the results were influenced by the way the teacher interpreted the instructions and suggestions and the way that they were applied.

Although the activities were arranged in terms of DCAT levels, there was no mathematical test for the concept of "easier" or "simpler" in this study. This means that if the students simplified a trigonometric expression, there was no mathematical test that could determine if the "simplified" trigonometric expression was indeed easier than the one that they had started with. Instead, this study used the lexical meaning of "easier".

The next section explores directions for future research, which is an extension of this study's limitations.

6.9 Recommendations for Future Research

In this section, this study presents possible directions for future research based on its findings. Further research could address the issues raised under limitations – such as repeating the study with a bigger sample and more schools and students.

The effect of teacher noticing per se in supporting students' productive struggles was not examined during this study. Hence, the effect of teacher noticing in supporting students' productive struggles is a recommendation for further study.

A more diverse group of students taught by more than one teacher at different schools might expose problems that were not evident in a smaller sample. It would be interesting to juxtapose the results of such a study with this study’s results. Moreover, to examine the long-term effect of productive struggles, a study lasting longer than six months could be carried out.

In the context of the United States of America, further research of particular interest could be on the influence of using productive struggles on African-American students. In his article on how to increase minority achievement in science and mathematics, Hrabowski III (2003) advised “stimulating the interest of minority students in math and science and preparing them to succeed academically in these fields” (p.44). Moreover, basing his argument on the 2015 Trends in International Mathematics and Science Study, Murphy (2019) reasoned that those students who attend schools with students from a higher socio-economic background achieve better mathematics results.

However, there is a great deal to learn from the class environment in this study, primarily because very little research exists on productive struggles in trigonometry.

Specific further research could include research on pre-service teachers’ productive struggles with trigonometry concepts, since they too consider trigonometry concepts to be challenging. In their investigation into pre-service teachers’ perceptions and knowledge of trigonometry, Nabie et al. (2018) found that pre-service teachers found trigonometry difficult, abstract and uninteresting to learn.

Next, this study expresses some recommendations on how to support students’ productive struggles.

6.10 Recommendations for supporting students' productive struggles

Some of the findings from this study would suggest that there are advantages for students using productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities. However, in this study one of the recurring mathematical shortcomings of students in the simplification of trigonometric expressions and the proving of trigonometric identities was in the areas of factorisation and operations on algebraic fractions. This study agrees with Amidon et al. (2020), Murdoch et al. (2020) and Vazquez et al. (2020), who contended that productive struggles expose students' knowledge deficiencies, especially when students have to apply previously learnt knowledge in new mathematical contexts. This study also observed that the students in this study compartmentalised knowledge. As such, this study agrees with Murdoch et al. (2020) that productive struggles should be a natural part of learning mathematics, especially in learning how to use prior knowledge in new contexts.

Direct instruction has many advantages in mathematics education. Flynn et al. (2012) and Gurses et al. (2015) argue that it is one of the best teaching strategies for improving student knowledge. However, when students in this study simplified trigonometric expressions and proved trigonometric identities, the influence of the class on a struggling student's ability to resolve a question seemed to be beneficial as well. Also, the mood in peer-to-peer instruction seemed to be more accepting of new ideas for problem solving.

Nevertheless, this study agrees with Choi et al. (2014) that a disadvantage of direct instruction is that students might have diminished problem-solving ability. Zhong et al. (2010) defines "problem-solving skills" as students' ability to investigate a solution to a problem or find a way to solve the problem. Furthermore, problem-solving involves teamwork and creative thinking (Siswono, 2014; Sockalingam, 2010). In this regard, the video analyses of this study suggest that there are advantages to students helping one another. In addition, the teacher's facilitation and the timing thereof are critical. There is still much to learn from Rowe's (1986) suggestion of "wait time" of more than three seconds by teachers when responding to a student's struggle. To overcome the inadequacies of this short and sometimes ineffective wait time, the trigonometric content knowledge of the teacher and pedagogical application thereof should be spot on. However, in this study the effective intervention of the teacher to assist students during their productive struggles, in collaboration with the students helping one another, was shown to be most effective.

It is imperative for the teacher to make sure that the students have the necessary prior knowledge, in order for the teacher to guide the students into problem-solving approaches. In this case it is necessary for the teacher to “fill in the gaps” with the necessary prior knowledge.

Although this study used Boaler and Brodie’s (2004) question types to analyse teacher questioning, each individual student may also require “on the fly” questioning by the teacher to best address their productive struggles, although this option does necessitate expansive content knowledge on the part of the teacher. In employing productive struggle, teachers can use the findings of this study – especially the students’ shortcomings – to adapt their lessons to incorporate these shortcomings of the students in new mathematical contexts such as the simplification of trigonometric expressions and the proving of trigonometric identities.

The conclusion of this study is presented next.

6.11 Conclusion

To analyse the productive struggles of students during the activity questions, this study used Warshauer’s (2011) framework for productive struggles, the framework of Kilpatrick et al. (2001) for mathematical proficiency, Schoenfeld’s (2018) teaching for robust understanding, Jacobs’ (2011) teacher noticing framework, and Boaler and Brodie’s (2004) proposed nine types of teacher questions. The ATD by Chevallard (1992) was the main theoretical framework.

The NEA framework was used to establish whether or not there was a link between the errors that the students made when they simplified trigonometric expressions and proved trigonometric identities on the assessments, and the productive struggles students experienced during their real-time DCAT learning activities.

Each of these frameworks mentioned complements the others in addressing the research questions. For example, using Warshauer’s (2011) productive struggle framework, this study could address the primary research question. In addition, the teacher questioning and noticing framework could uncover any mathematical proficiency, or lack thereof by the students. However, taken together these frameworks target how students move through productive struggles when they simplify trigonometric expressions and prove trigonometric identities.

The advantage of analysing the research questions through this collection of frameworks is to understand the different students' productive struggles during the simplification of trigonometric expressions and the proving of trigonometric identities. For example, in collaboration with teacher *noticing* and *questioning,* the students in the class also helped to resolve some of the struggling students' impasses. Moreover, this study used the mathematical proficiency framework of Kilpatrick et al. (2001) to provide the mathematical proficiencies the students lacked that caused them to struggle, for example with the manipulation of algebraic fractions in the simplification of trigonometric expressions. In addition, Schoenfeld's (2018) teaching for robust understanding framework was used to give a didactic perspective on students' productive struggles in the simplification of trigonometric expressions and the proving of trigonometric identities.

Each of the frameworks mentioned above was analysed using a qualitative research methodology, supported by descriptive statistics. Through qualitative research methods this study could identify the types of productive struggles that were observed during the simplification of trigonometric expressions and the proving of trigonometric identities. That is, a qualitative research methodology gave this study the ability to distinguish between, for example, "getting started" and "misconceptions and errors" productive struggles. On the other hand, descriptive statistics was used to describe in words what was presented in a table. That is, through descriptive statistics this study could relate a number to an action of productive struggle.

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APPENDICES

Appendix 1: Consent Letter – District Superintendent

DEPARTMENT OF APPLIED MATHEMATICS

UNIVERSITY OF CAPE TOWN
PRIVATE BAG X3
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SOUTH AFRICA

RESEARCHER: Anthony Sayster
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Informed Voluntary Consent to Participate in Research Study
Project Title: Promoting strategic learning by exploring students' productive struggle in trigonometry

Dear Sir

In collaboration with the University of Cape Town (RSA) I am embarking on an exciting endeavour in Mathematics Education research. Productive Struggles in Mathematics is a relatively new research topic in which I am currently reading for my PhD. in mathematics education.
Chesterfield High School in Chesterfield County School District (CCSD) was chosen for its excellent teachers and well-behaved students to conduct the study. A different teacher will be conducting the study to ensure impartiality and objectivity of the results. The results will be presented at an international conference giving our school district and county much needed positive international exposure.

Procedures
During the study which will last about 7 weeks, students will be observed through voice- and video-recording how they tackle difficult mathematical problems. Furthermore, students' participation in the study is not obligatory, and students can withdraw at any point during the study. Confidentiality and anonymity will be guaranteed. Since the students are minors, I will seek permission from their parents to participate in the study.

Student Work: Worksheets of student work may be collected for data analysis. If you object to this, please indicate this below.

I agree that worksheets of student work may be collected and used for this research:

Permission Granted: YES ☐ ☐ **NO (Initials)**

While video- and audio-recording will be taken, this information will be kept entirely private. In any publications or presentations, all participants will be anonymized through blurring where necessary.
Recording: I may take photographs, audio-record audio and/or video-record as part of the data collection of the study. If you object to this, please indicate this below.
I agree that the participating students at Chesterfield High School may be photographed/audio-recorded/video-recorded (strikethrough as applicable)

☐ **YES** ☐ **NO (initials)**

I agree to the use of properly anonymized photographs/ audio recordings/ videos in websites and publications of the participating students at Chesterfield High School for research purposes (strikethrough as applicable)

☐ **YES** ☐ **NO (initials)**

Disclaimer/Withdrawal:
Students' participation is completely voluntary; students may refuse to participate, and they may withdraw from the study at any time without having to state a reason. Withdrawal by students from participating in the study will not result in any prejudice or penalty against them. Should a student choose to withdraw, the I commit not to use any of the information students have provided prior withdrawal. Note that the I may also withdraw student from the study at any time.
Confidentiality:
All collected data will be kept in a locked cabinet and will only be used for the purposes of the proposed study. Transcribed and coded data will be saved on an encrypted storage device. The participants will also be informed about the data management policy at the University of Cape Town which relates to the notion that all research data will be kept in data repository sites for future use.
Risks and Benefits: There are no harmful risks to Chesterfield County School District related to its participation in the research study. The results of the study will be presented at an international conference and thus our school district and county may contribute to the implementation of learning strategies internationally.

It is hereby that I request permission from you to allow me to carry out my research at Chesterfield High School which is in Chesterfield County.

What signing this form means:
By signing this consent form, you agree to allow me to carry out my PhD research at Chesterfield High School. The aim, procedures to be used, as well as the potential risks and benefits have been explained verbally to you in detail, using this form. You are free to contact me, to ask questions or request further information, at any time during this research.

Regards.

Anthony Sayster
E-mail: asayster@chesterfieldschools.org
Phone: +1 843 623 2161

I agree that the research can be done at Chesterfield High School in CCSD:

Permission Granted: ☐ **Yes** ☐ **No (Initials)**

Superintendent's Name Printed: ____

Superintendent's Signature: ____

Appendix 2: Consent Letter – Headmaster

DEPARTMENT OF APPLIED MATHEMATICS

UNIVERSITY OF CAPE TOWN
PRIVATE BAG X3
RONDEBOSCH 7701
SOUTH AFRICA

RESEARCHER: Anthony Sayster
TELEPHONE: 1-843-662-9662
E-MAIL: asayster@chesterfieldschools.org
URL:

Informed Voluntary Consent to Participate in Research Study
Project Title: Promoting strategic learning by exploring students' productive struggle in trigonometry

Dear Sir
In collaboration with the University of Cape Town (RSA) I am embarking on an exciting endeavor in Mathematics Education research. Productive Struggles in Mathematics is a relatively new research topic in which I am currently reading for my PhD.
Chesterfield High School was chosen for its excellent teachers and well-behaved students to conduct the study. A different teacher will be conducting the study to ensure impartiality and objectivity of the results. The results will be presented at an international conference giving our school district and county much needed positive international exposure.
Procedures
During the study which will last about 7 weeks, students will be observed through voice and video recordings on how they tackle difficult mathematical problems. Furthermore, students' participation in the study is not obligatory and students can withdraw at any point during the study. Since the students are minors, I will seek permission from their parents to participate.
Student Work: Worksheets of student work may be collected for data analysis. If you object to this, please indicate this below.

I agree that student worksheets may be collected and used for this research:

Permission Granted: ☐ **YES** ☐ **NO (Initials)**

While video- and audio- recording will be taken, this information will be kept entirely private. In any publications or presentations, all participants will be anonymized through blurring where necessary
Recording: I may take photographs, audio- record, or video- record as part of the study. If you object to this, please indicate this below.

I agree that the participating students may be photographed/audio-recorded/video-recorded (strike through as applicable)

☐ **YES** ☐ **NO (initials)**

I agree to the use of properly anonymized photographs/ audio recordings/ videos in websites and publications of the participating students for research purposes (strike through as applicable)

☐ **YES** ☐ **NO (initials)**

Disclaimer/Withdrawal:
Students' participation is completely voluntary; students may refuse to participate. and students may withdraw at any time without having to state a reason. Students' who withdraw from the study will not be prejudiced or penalized in any way. Should any student choose to withdraw, I commit not to use any of the information they have provided prior to their withdrawal. Note that I may also withdraw a student from the study at any time.

All collected data will be kept in a locked cabinet and will only be used for the purposes of the proposed study. Transcribed and coded data will be saved on an encrypted storage device. The students will also be informed about the data management policy at the University of Cape Town which relates to the notion that all research data will be kept in data repository sites for future use.

Confidentiality:
All collected data will be kept in a locked cabinet and will only be used for the purposes of the proposed study. Transcribed and coded data will be saved on an encrypted storage device. The participants will also be informed about the data management policy at the University of Cape Town which relates to the notion that all research data will be kept in data repository sites for future use.

Risks and Benefits: There are no harmful risks to the participating students in the study. The results of the study will be presented at an international conference and thus our school district and county may contribute to the implementation of learning strategies internationally.

It is hereby that I request permission from you, the headmaster, to conduct the study at Chesterfield High School during the 2018-2019 school year.

What signing this form means:
By signing this consent form, you agree to give me permission to carry out my research study at Chesterfield High School. The aims, procedures to be used, as well as the potential risks, and benefits of your school's participation have been explained verbally to you in detail, using this form. You are free to contact me, to ask questions or request further information, at any time during this research.

Regards.

Anthony Sayster
E-mail: asayster@chesterfieldschools.org
phone: 843 3374442

I agree that the school participate in this research:

Permission Granted: ☐ **Yes** ☐ **No (Initials)**

Principal's Name Printed: ____

Principal's Signature: ____

Appendix 3: Consent Letter – Participating Teacher

DEPARTMENT OF APPLIED MATHEMATICS

UNIVERSITY OF CAPE TOWN
PRIVATE BAG X3
RONDEBOSCH 7701
SOUTH AFRICA

RESEARCHER: Anthony Sayster
TELEPHONE: 1-843-662-9662
E-MAIL: asayster@chesterfieldschools.org
URL:

Informed Voluntary Consent to Participate in Research Study
Project Title: Promoting strategic learning by exploring students' productive struggle in trigonometry

Dear Sir/Ma'am

In collaboration with the University of Cape Town (RSA) I am embarking on an exciting endeavor in Mathematics Education research. Productive Struggles in Mathematics is a relatively new research topic in which I am currently reading for my PhD.

Chesterfield High School was chosen for its excellent teachers and well-behaved students to conduct the study. To ensure impartiality and objectivity of the results, I am requesting you to be the participating teacher for the research study. The results will be presented at an international conference giving our school district and county much needed positive international exposure.

Procedures
During the study which will last about 7 weeks, students will be observed through voice and video recording how they tackle difficult mathematical problems. Participants' engagement in the study is not obligatory and they can withdraw at any point during the study.
Student Work: Worksheets of student work may be collected for data analysis. If you object to this, please indicate this below.

I agree to collect student worksheets for this research:

Permission Granted: ☐ **YES** ☐ **NO (Initials)**

While video and audio recording will be taken, this information will be kept entirely private. In any publications or presentations, all participants including the participating teacher will be anonymized through blurring where necessary.

Recording: I may take photographs, audio-record audio and/or video-record as part of the data collection of the study. If you object to this, please indicate this below.

I agree that I may be photographed/audio-recorded/video-recorded (strikethrough as applicable)

☐ **YES** ☐ **NO (initials)**

I agree to the use of properly anonymized photographs/ audio recordings/ videos in websites and publications of myself for research purposes (strikethrough as applicable)

☐ **YES** ☐ **NO (initials)**

Confidentiality:

All collected data will be kept in a locked cabinet and will only be used for the purposes of the proposed study. Transcribed and coded data will be saved on an encrypted storage device. The participants will also be informed about the data management policy at the University of Cape Town which relates to the notion that all research data will be kept in data repository sites for future use.

Risks and Benefits: There are no harmful risks to you, related to your participation in this study. The results of the study will be presented at an international conference and thus our school district and county may contribute to the implementation of learning strategies internationally.

What signing this form means:
By signing this consent form, you agree to participate in this research study as the participating teacher. The aim, procedures to be used, as well as the potential risks and benefits of your participation have been explained verbally to you in detail, using this form. Refusal to participate in or withdrawal from this study at any time will have no effect on you in any way. You are free to contact me, to ask questions or request further information, at any time during this research.

Regards

Anthony Sayster
E-mail: asayster@chesterfieldschools.org
Phone: +1 843 623 2161
I agree to be the participating teacher in this research:

Permission Granted: ☐ **YES** ☐ **NO (initials)**

Participating Teacher's Name Printed: ____

Participating Teacher's Signature: ____

Appendix 4: Consent Letter – Parents

DEPARTMENT OF APPLIED MATHEMATICS

UNIVERSITY OF CAPE TOWN
PRIVATE BAG X3
RONDEBOSCH 7701
SOUTH AFRICA

RESEARCHER: Anthony Sayster
TELEPHONE: 1-843-662-9662
E-MAIL: asayster@chesterfieldschools.org
URL:

Informed Voluntary Consent to Participate in Research Study
Project Title: Promoting strategic learning by exploring students' productive struggle in trigonometry

Dear Sir/Ma'am

In collaboration with the University of Cape Town (RSA) I am embarking on an exciting endeavour in Mathematics Education research. Productive Struggles in Mathematics is a relatively new research topic in which I am currently reading for my PhD in mathematics education.
Chesterfield High School was chosen for its excellent teachers and well-behaved students to conduct the study. Mr. Michael Maines will be the teacher conducting the study to ensure impartiality and objectivity of the results.

Procedures
During the study which will last about 7 weeks, students will be observed through voice- and video-recording as they tackle difficult mathematical problems. Furthermore, students' participation in the study is not obligatory and they can withdraw at any point during the study. Confidentiality and anonymity will be guaranteed. Given that your child is a minor, his/her participation on the study requires your consent.

Student Work: Worksheets of student work may be collected for data analysis. If you object to this, please indicate this below.

I agree that student worksheets may be collected and used for this research:

Permission Granted: ☐ **YES** ☐ **NO (Initials)**

While video and audio recording will be taken, this information will be kept entirely private. In any publications or presentations, all participants will be anonymized through blurring where necessary.
Recording: I may take photographs, audio-record audio and/or video-record as part of the data collection of the study. If you object to this, please indicate this below.

I agree that my child may be photographed/audio-recorded/video-recorded (strikethrough as applicable)

☐ **YES** ☐ **NO (initials)**

I agree to the use of properly anonymized photographs/ audio recordings/ videos in websites and publications of my child for research purposes (strikethrough as applicable)

☐ **YES** ☐ **NO (initials)**

Disclaimer/Withdrawal: Your child's participation is completely voluntary; she/he may refuse to participate, and she/he may withdraw at any time without having to state a reason. Withdrawal of participation by your child will not lead to any form of prejudice or penalty against her/him. Should your child choose to withdraw, then I commit not to use any of the information she/he has provided prior to withdrawal. Note that I may also withdraw your child from the study at any time.

Confidentiality:
All collected data will be kept in a locked cabinet and will only be used for the purposes of the proposed study. Transcribed and coded data will be saved on an encrypted storage device. The participants will also be informed about the data management policy at the University of Cape Town which relates to the notion that all research data will be kept in data repository sites for future use.

It is hereby that I request permission from you to allow your child to participate in the study.

Risks and Benefits: There are no harmful risks to your child related to participation in this study. The results of the study will be presented at an international conference and thus our school district and county may contribute to the implementation of learning strategies internationally.

What signing this form means:
By signing this consent form, you agree to allow your child to participate in this study. The aim, procedures to be used, as well as the potential risks and benefits of your child's participation have been explained verbally to you in detail, using this form. You are free to contact me, to ask questions or request further information, at any time during this research.

Regards

Anthony Sayster
E-mail: asayster@chesterfieldschools.org
Phone: 843 623 2161

Student's Name: ____

I agree that my child may participate in this research:

Permission Granted: ☐ **Yes** ☐ **No (Initials)**

Parent's Name Printed: ____

Parent's Signature: ____

Appendix 5: Clearance certificate

UNIVERSITY OF CAPE TOWN
IYUNIVESITHI YASEKAPA • UNIVERSITEIT VAN KAAPSTAD

Faculty of Science
University of Cape Town
Rondebosch
South Africa 7701

Tel: +27 21 650 2866/7
E-mail: Rachel.Wynberg@uct.ac.za

12 March 2019

Mr Anthony Sayster
Department of Applied Mathematics

RE: Promoting strategic learning by exploring students' productive struggle in trigonometry.

Dear Mr Anthony Sayster

I am pleased to inform you that the Faculty of Science Research Ethics Committee has approved the above-named application for research ethics clearance, subject to the conditions listed below.

- Implement the measures described in your application to ensure that the process of your research is ethically sound; and
- Uphold ethical principles throughout all stages of the research, responding appropriately to unanticipated issues: please contact me if you need advice on ethical issues that arise.

Your approval code is: **FSREC 21 - 2019**

I wish you success in your research.

Yours sincerely

A/Prof Rachel Wynberg
Chair: Faculty of Science Research Ethics Committee

Cc: Dr Jonathan Stock (Supersvisor)

Appendix 6: DCAT 1 Activity 1 Questions

Simplifying the following trigonometric expressions:

Day one:

Time: Approximately 40 min

Question 1

Simplify the trigonometric expression $sinx\frac{1}{tanx}$.

Solution:

$sinx\frac{1}{tanx}$

$= \frac{sinx}{1}\frac{1}{\frac{sinx}{cosx}}$

$= \frac{sinx}{1}\frac{cosx}{sinx}$

$= cosx.$

Question 2

Simplify the trigonometric expression $sin^2x + cos^2x + 1$.

Solution:

$sin^2x + cos^2x + 1$

$= 1 + 1$

$= 2.$

Question 3

Simplify the trigonometric expression $sinxtanxcosx$.

Solution:

$sinxtanxcosx$

$= \frac{sinx}{1}\frac{sinx}{cosx}\frac{cosx}{1}$

$= sin^2x.$

Question 4

Simplify the trigonometric expression $(\frac{cosx}{tanx})sinx$.

Solution:

$$\left(\frac{cosx}{tanx}\right) sinx$$

$$= \frac{cosx}{\frac{sinx}{cosx}} \frac{sinx}{1}$$

$$= \frac{cosx}{1} \frac{cosx}{sinx} \frac{sinx}{1}$$

$$= cos^2x.$$

Question 5

Simplify the trigonometric expression $\frac{sinx+cosx}{sinxcosx}$.

Solution:

$$\frac{sinx + cosx}{sinxcosx}$$

$$= \frac{sinx}{sinxcosx} + \frac{cosx}{sinxcosx}$$

$$= \frac{1}{cosx} + \frac{1}{sinx}.$$

Day two

Question 1

Simplify the trigonometric expression $\frac{sinx}{tanx}$.

Solution:

$$\frac{sinx}{tanx}$$

$$= \frac{sinx}{\frac{sinx}{cosx}}$$

$$= \frac{sinx}{1} \frac{cosx}{sinx}$$

$$= cosx.$$

Question 2

Simplify the trigonometric expression $(sinx + cosx)^2 - 2sinxcosx$.

Solution:

$$(sinx + cosx)^2 - 2sinxcosx$$

$$= sin^2x + 2sinxcosx + cos^2x - 2sinxcosx$$

$= sin^2x + cos^2x$

$= 1.$

Question 3

Simplify the trigonometric expression tan^2xcos^2x.

Solution:

tan^2xcos^2x

$= \frac{sin^2x}{cos^2x}\frac{cos^2x}{1}$

$= sin^2x.$

Question 4

Simplify the trigonometric expression $tan^2x + 1$.

Solution:

$\frac{tan^2x}{1} + 1$

$= \frac{sin^2x}{cos^2x} + 1$

$= \frac{sin^2x}{cos^2x} + \frac{cos^2x}{cos^2x}$

$= \frac{sin^2x + cos^2x}{cos^2x}$

$= \frac{1}{cos^2x}.$

Question 5

Simplify the trigonometric expression $\frac{cosx}{1 - sin^2x}$

Solution:

$\frac{cosx}{1 - sin^2x}$

$= \frac{cosx}{cos^2x}$

$= \frac{1}{cosx}.$

Assessment One:

Time: 45 min.

Show ALL your work. No work, no credit.

Simplify the following trigonometric expressions.

1. $\frac{cosx}{\frac{1}{tanx}}$
2. $\frac{sinx}{1-cos^2x}$
3. $\frac{sin^2x+cos^2x}{2sinx}$
4. $\frac{cos^2x}{1-cos^2x}$
5. $\frac{1}{cos^2x} - 1$
6. $\frac{1}{tan^2x} sin^2x$
7. $tanxcosx + sinx$

Solutions:

1. $\frac{cosx}{\frac{1}{tanx}}$

$= \frac{cosx}{1} \frac{tanx}{1}$

$= \frac{cosx}{1} \frac{sinx}{cosx}$

$= sinx.$

2. $\frac{sinx}{1-cos^2x}$

$= \frac{sinx}{sin^2x}$

$= \frac{1}{snx}.$

3. $\frac{sin^2x+cos^2x}{2sinx}$

$= \frac{1}{2sinx}.$

4. $\frac{cos^2x}{1-cos^2x}$

$= \frac{cos^2x}{sin^2x}$

$= \frac{1}{\frac{sin^2x}{cos^2x}}$

$= \frac{1}{tan^2x}.$

5. $\frac{1}{cos^2x} - 1$

$= \frac{1}{cos^2x} - \frac{cos^2x}{cos^2x}$

$= \frac{1-cos^2x}{cos^2x}$

$= \frac{sin^2x}{cos^2x}$

$= tan^2x.$

6. $\frac{1}{tan^2x} sin^2x$

$= \frac{cos^2x}{sin^2x} sin^2x$

$= cos^2x.$

7. $tanxcosx + sinx$

$= \frac{sinx}{cosx} cosx + sinx$

$= sinx + sinx$

$= 2sinx.$

Appendix 7: DCAT 2 Activity 1 Questions

Simplifying the following trigonometric expressions:

Day one:

Time: Approximately 40 minutes

Question 1

Simplify the trigonometric expression $\frac{sinx}{cos^3x} \div \frac{tanx}{cos^3x}$.

Solution:

$\frac{sinx}{cos^3x} \div \frac{tanx}{cos^3x}$

$= \frac{sinx}{cos^3x} * \frac{cos^3x}{tanx}$

$= \frac{sinx}{tanx}.$

$= \frac{sinx}{\frac{sinx}{cosx}}$

$= \frac{sinx}{1} \frac{cos}{sinx}$

$= cosx.$

Question 2

Simplify the trigonometric expression $sinxcos^2x - sinx.$

Solution:

$sinxcos^2x - sinx$

$= sinx(cos^2x - 1)$

$= sinx(-sin^2x)$

$= -sin^3x.$

Question 3

Simplify the trigonometric expression $\frac{2cosx}{sinx} + \frac{2sinx}{cosx}.$

Solution:

$\frac{2cosx}{sinx} + \frac{2sinx}{cosx}$

$= \frac{2cosx(cosx)}{sinx(cosx)} + \frac{2sinx(sinx)}{cosx(sinx)}$

$= \frac{2cos^2+2sin^2x}{sinxcosx}$

$= \frac{2(sin^2x+cos^2x)}{sinxcosx}$

$= \frac{2*1}{sinxcosx}$

$= \frac{2}{sinxcosx}.$

Question 4

Simplify the trigonometric expression $\frac{cos^2x}{1-sinx}.$

Solution:

$\frac{cos^2x}{1 - sinx}$

$= \frac{1-sin^2x}{1-sinx}$

$= \frac{(1-sinx)(1+sinx)}{(1-sinx)}$

$= 1 + sinx.$

Question 5

Simplify the trigonometric expression $sinx + \frac{cosx}{tanx}$.

Solution:

$sinx + \frac{cosx}{tanx}$

$= \frac{sinx}{1} + \frac{cosx}{\frac{sinx}{cosx}}$

$= \frac{sinx}{1} + \frac{cosx}{1}\frac{cosx}{sinx}$

$= \frac{sin^2x}{sinx} + \frac{cos^2x}{sinx}$

$= \frac{sin^2x+cos^2x}{sinx}$

$= \frac{1}{sinx}.$

Day two

Time: Approximately 40 minutes.

Question 1

Simply the trigonometric expression $sinxcosxtanx + cos^2x$.

Solution:

$sinxcosxtanx + cos^2x$

$= \frac{sinxcosx}{1}\frac{sinx}{cosx} + cos^2x$

$= sin^2x + cos^2x$

$= 1.$

Question 2

Simplify the trigonometric expression $\frac{1}{sin^2x} + \frac{1}{cos^2x}$.

Solution: $\frac{1}{sin^2x} + \frac{1}{cos^2x}$

$= \frac{cos^2x}{sin^2xcos^2x} + \frac{sin^2x}{cos^2xsin^2x}$

$= \frac{cos^2x+sin^2x}{sin^2xcos^2x}$

$= \frac{1}{sin^2xcos^2x}$.

Question 3

Simplify the trigonometric expression $\frac{sin^2x}{1+cosx}$.

Solution:

$\frac{sin^2x}{1+cosx}$

$= \frac{1-cos^2x}{1+cosx}$

$= \frac{(1+cosx)(1-cosx)}{1+cosx}$

$= 1 - cosx.$

Question 4

Simplify the trigonometric expression $\frac{1}{tan^2x+1}$.

Solution: $\frac{1}{tan^2x+1}$

$= \frac{1}{\frac{sin^2x}{cos^2x}+\frac{cos^2x}{cos^2x}}$

$= \frac{1}{\frac{sin^2x+cos^2x}{cos^2x}}$

$= \frac{1}{\frac{1}{cos^2x}}$

$= \frac{cos^2x}{1}$

$= cos^2.$

Question 5

Simply the trigonometric expression $\frac{sin^2x+3sinx+2}{sin^2x+4sinx+3}$.

Solution:

$$\frac{sin^2x + 3sinx + 2}{sin^2x + 4sinx + 3}$$
$$= \frac{(2+sinx)(1+sinx)}{(3+sinx)(1+sinx)}$$
$$= \frac{2+sinx}{3+sinx}.$$

The teacher should stress the reason it is preferable to write $2 + sinx$ instead of $sinx + 2$.

<u>Assessment Two:</u>

Time: 45 min.

Show ALL your work. No work, no credit.

Simplify the following trigonometric expressions.

1. $1 + tan^2x$
2. $tan^2x - tan^2sin^2x$
3. $sin^2x + \frac{cosxsinx}{tanx}$
4. $\frac{1}{tanx} + \frac{1}{sinx}$
5. $\frac{cos^2x}{sin^2x}\frac{tan^2x}{1}$
6. $\frac{2sin^2x+5sinx-3}{3+sinx}$
7. $\frac{sin^2x-1}{sin^2x+sinx-2}$

Solutions:

1. $1 + tan^2x$

$$= 1 + \frac{sin^2x}{cos^2x}$$
$$= \frac{cos^2x}{cos^2x} + \frac{sin^2x}{cos^2x}$$
$$= \frac{cos^2x+sin^2x}{cos^2x}$$
$$= \frac{1}{cos^2x}.$$

2. $tan^2x - tan^2xsin^2x$

$$= tan^2x(1 - sin^2x)$$
$$= tan^2x(cos^2x)$$
$$= \frac{sin^2x}{cos^2x}\frac{(cos^2x)}{1}$$

$= sin^2x.$

3. $sin^2x + \frac{cosxsinx}{tanx}$

$= sin^2x + \frac{cosxsinx}{\frac{sinx}{cosx}}$

$= sin^2x + \frac{cosxsinx}{1}\frac{cosx}{sinx}$

$= sin^2x + cos^2x$

$= 1.$

4. $\frac{1}{tanx} + \frac{1}{sinx}$

$= \frac{1}{\frac{sinx}{cosx}} + \frac{1}{sinx}$

$= \frac{cosx}{sinx} + \frac{1}{sinx}$

$= \frac{1+cosx}{sinx}.$

5. $\frac{cos^2x}{sin^2x}\frac{tan^2x}{1}$

$= \frac{cos^2x}{sin^2x}\frac{sin^2x}{cos^2x}$

$= 1.$

6. $\frac{2sin^2x+5sinx-3}{3+sinx}$

$= \frac{(-1+2sinx)(3+sinx)}{(3+sinx)}$

$= -1 + 2sinx.$

7. $\frac{sin^2x-1}{sin^2x+sinx-2}$

$= \frac{(sinx+1)(sinx-1)}{(sinx-1)(sinx+2)}$

$= \frac{sinx+1}{sinx+2}.$

Appendix 8: DCAT 3 Activity 1 Questions

Simplifying the following trigonometric expressions:

Day one:

Time: Approximately 40 min

Question 1

Simplify the trigonometric expression $tan^2x - tan^2xsin^2x$.

Solution:

$tan^2x - tan^2xsin^2x$

$= tan^2x(1 - sin^2x)$

$= tan^2xcos^2x$

$= \frac{sin^2x}{cos^2x}\frac{cos^2x}{1}$

$= sin^2x.$

Question 2

Simplify the trigonometric expression $\frac{2sin^2x - sinx - 1}{sinx - 1}$.

Solution:

$\frac{2sin^2x - sinx - 1}{sinx - 1}$

$= \frac{(2sinx+1)(sinx-1)}{sinx-1}$

$= 2sinx + 1.$

Question 3

Simplify the trigonometric expression $tanx + \frac{cosx}{1+sinx}$.

Solution:

$tanx + \frac{cosx}{1 + sinx}$

$= \frac{sinx}{cosx} + \frac{cosx}{1+sinx}$

$= \frac{sinx(1+sinx)}{cosx(1+sinx)} + \frac{cosx * cosx}{(1+sinx)cosx}$

$= \frac{sinx+sin^2x+cos^2x}{(1+sinx)(cosx)}$

$= \frac{1+sinx}{(1+sinx)cosx}$

$= \frac{1}{cosx}.$

Question 4

Simplify the trigonometric expression $\frac{sinx-cosx}{sinx+cosx}+\frac{sinx+cosx}{sinx-cosx}$.

Solution:

$$\frac{sinx-cosx}{sinx+cosx}+\frac{sinx+cosx}{sinx-cosx}$$

$$=\frac{(sinx-cosx)(sinx-cosx)}{(sinx+cosx)(sinx-cosx)}+\frac{(sinx+cosx)(sinx+cosx)}{(sinx-cosx)(sinx+cosx)}$$

$$=\frac{sin^2x-2sinxcosx+cos^2x+sin^2x+2sinxcosx+cos^2x}{(sinx-cosx)(sinx+cosx)}$$

$$=\frac{2sin^2x+2cos^2x}{(sinx-cosx)(sinx+cosx)}$$

$$=\frac{2(sin^2x+cos^2x)}{(sinx-cosx)(sinx+cosx)}$$

$$=\frac{2*1}{sin^2x-cos^2x}$$

$$=\frac{2}{sin^2x-cos^2x}.$$

Day two

Simplifying trigonometric expressions

Time: Approximately 40 minutes

Question 1

Simplify the trigonometric expression $\frac{sin^3x+cos^3x}{sinx+cosx}$

Solution:

$$\frac{sin^3x+cos^3x}{sinx+cosx}$$

$$=\frac{(sinx+cosx)(sin^2x-sinxcosx+cos^2x)}{(sinx+cosx)}$$

$$=\frac{(sinx+cosx)(1-sinxcosx)}{(sinx+cosx)}$$

$$=1-sinxcosx.$$

Question 2

Simplify the trigonometric expression $\frac{2sinx+4}{3sin^2x+7sinx+2}$.

Solution:

$$\frac{2sinx+4}{3sin^2x+7sinx+2}$$

$$=\frac{2(sinx+2)}{(3sinx+1)(sinx+2)}$$

$$=\frac{2}{3sinx+1}.$$

Question 3

Simplify the trigonometric expression $\frac{1}{\sin^2 x-1}+\frac{\sin x}{\sin^2 x-1}$.

Solution:

$$\frac{1}{sin^2x-1}+\frac{sinx}{sin^2x-1}$$

$=\frac{1+sinx}{sin^2x-1}$

$=\frac{1+sinx}{(sinx+1)(sinx-1)}$

$=\frac{1}{sinx-1}$.

Question 4

Simplify the trigonometric expression $\frac{1-cos^2x}{sin^2x}-\frac{1-sin^2x}{cos^2x}$.

Solution:

$$\frac{1-cos^2x}{sin^2x}-\frac{1-sin^2x}{cos^2x}$$

$=\frac{sin^2x}{sin^2x}-\frac{cos^2x}{cos^2x}$

$= 1-1= 0$

Assessment Three

Simplify the trigonometric expressions. Show all your work.

Time: 45 minutes.

1. $sin^2x+cos^2x*tan^2x$
2. $\frac{1}{tanx}+\frac{sinx}{1+cosx}$
3. $\frac{1+tanx}{1-tanx}-\frac{1-tanx}{1+tanx}$
4. $\frac{2sinx+1}{2sin^2x+5sinx-3}$
5. $\frac{cosx}{1-cos^2x}+\frac{1}{1-cos^2x}$

Solutions:

1. $sin^2x+cos^2x*tan^2x$

   $=sin^2x+cos^2x\frac{sin^2x}{cos^2x}$

   $= sin^2x+sin^2x$

$$= 2sin^2x$$

2. $\frac{cosx}{sinx} + \frac{sinx}{1+cosx}$

$$= \frac{Cosx(1+cosx)}{sinx(1+cosx)} + \frac{sinxsinx}{sinx(1+cosx)}$$

$$= \frac{cosx+cos^2x+sin^2x}{sinx(1+cosx)}$$

$$= \frac{cosx+1}{sinx(1+cosx)}$$

$$= \frac{1}{sinx}$$

3. $\frac{1+tanx}{1-tanx} - \frac{1-tanx}{1+tanx}$

$$= \frac{(1+tanx)(1+tanx)}{(1-tanx)(1+tanx)} - \frac{(1-tanx)(1-tanx)}{(1+tanx)(1-tanx)}$$

$$= \frac{1+2tanx+tan^2x}{(1+tanx)(1-tanx)} - \frac{1-2tanx+tan^2x}{(1+tanx)(1-tanx)}$$

$$= \frac{1+2tanx+tan^2x-1+2tanx-tan^2x}{(1+tanx)(1-tanx)}$$

$$= \frac{4tanx}{1-tan^2x}$$

4. $\frac{2sinx+1}{2sin^2x+5sinx-3}$

$$= \frac{2sinx+1}{(2sinx-1)(sinx+3)}$$

5. $\frac{cosx}{1-cos^2x} + \frac{1}{1-cos^2x}$

$$= \frac{cosx+1}{1-cos^2x}$$

$$= \frac{cosx+1}{(1-cosx)(1+cosx)}$$

$$= \frac{1}{1-cosx}$$

Appendix 9: DCAT 1 Activity 2 Questions

Prove the following trigonometric identities:

Day one:

Time: Approximately 40 min

Question 1

Prove that $cosxtanx = sinx$

Solution:

$cosxtanx$

$= \frac{cosx}{1}\frac{sinx}{cosx}$

$= sinx.$

Question 2

Prove that $1 - sin^2x + cos^2x = 2cos^2x.$

Solution:

$1 - sin^2x + cos^2x$

$= cos^2x + cos^2x$

$= 2cos^2x.$

Question 3

Prove that $\frac{\frac{1}{cosx}}{\frac{1}{sinx}} = tanx.$

Solution:

$\frac{\frac{1}{cosx}}{\frac{1}{sinx}}$

$= \frac{1}{cosx}\frac{sinx}{1}$

$= \frac{sinx}{cosx}$

$= tanx.$

Question 4

Prove that $16sin^2xcosx + 12cos^2xsinx = 4sinxcosx(4sinx + 3cosx).$

Solution:

$16sin^2xcosx + 12cos^2xsinx$

$= 4sinxcosx(4sinx + 3cosx).$

Where $4sinxcosx$ is the Highest Common Factor in the expression $16sin^2xcosx + 12cos^2xsinx.$

Question 5

Prove that $\frac{1}{sin^2x}\frac{1}{cos^2x} = \frac{1}{sin^2x} + \frac{1}{cos^2x}$.

Solution:

$$\frac{1}{sin^2x} + \frac{1}{cos^2x}$$

$$= \frac{sin^2x + cos^2x}{sin^2x cos^2x}$$

$$= \frac{1}{sin^2x cos^2x}$$

$$= \frac{1}{sin^2x}\frac{1}{cos^2x}.$$

Day two (Activity – groups of 2-3 students)

After illustrating how identities are proven the students will complete the following activity, where they logically arrange a proof of an identity by scrambling the steps. The teacher will cut out the steps for each question and scramble them, and hand them to the students to unscramble and arrange each proof.

Question 1

Prove that $tanx + \frac{2}{tanx} = \frac{sin^2x + 2cos^2x}{sinxcosx}$.

$$tanx + \frac{2}{tanx}$$

$$\frac{sin^2x}{sinxcosx} + \frac{2cos^2x}{sinxcosx}$$

$$\frac{sinx}{cosx} + 2\frac{cosx}{sinx}$$

Question 2

Prove that $tanx + \frac{1}{tanx} = \frac{1}{sinxcosx}$.

$$\frac{sin^2x + cos^2x}{sinxcosx}$$

$$\frac{\frac{sinx}{cosx} + \frac{1}{sinx}}{cosx}$$

$$\frac{1}{sinxcosx}$$

$$\frac{sinx}{cosx} + \frac{cosx}{sinx}$$

Question 3

Prove that $\frac{tanx - sinx}{sinx} = \frac{1}{cosx} - 1$.

$$\frac{\frac{sinx}{cosx}}{sinx} - \frac{sinx}{sinx}$$

$$\frac{1}{cosx} - 1$$

$$\frac{sinx}{cosx}\frac{1}{sinx} - 1$$

$$\frac{tanx}{sinx} - \frac{sinx}{sinx}$$

Question 4

Prove that $\frac{1}{cos^2x}(1 - cos^2x) = tan^2x$.

$$tan^2x$$

$$\frac{1 - cos^2x}{cos^2x}$$

$$\frac{sin^2x}{cos^2x}$$

$$\frac{1}{cos^2x} - \frac{cos^2x}{cos^2x}$$

Question 5

Prove the following identity $sinx - sinxcos^2x = sin^3x$.

$$sinxsin^2x$$

$$sin^3x$$

$$sinx(1 - cos^2x)$$

Assessment One:

Time: 45 min.

Show ALL your work. No work no credit

Prove the following trigonometric identities.

1. $\frac{tanx}{\frac{1}{cosx}} = sinx$
2. $\frac{\frac{1}{cosx}}{\frac{1}{sinx}} = tanx$
3. $tan^2xcos^2x = sin^2x$
4. $tan^2x = \frac{sin^2x}{1-sin^2x}$
5. $1+\frac{sinx}{cosx} = \frac{sinx+cosx}{cosx}$
6. $sin^2x + 2cos^2x = 1 + cos^2x$
7. $tanxsinx + cosx = \frac{1}{cosx}$

Solutions:

1. $\frac{tanx}{\frac{1}{cosx}}$

$= \frac{tanx}{1}\frac{cosx}{1}$

$= \frac{sinx}{cosx}\frac{cosx}{1}$

$= sinx$

2. $\frac{\frac{1}{cosx}}{\frac{1}{sinx}}$

$= \frac{1}{cosx}\frac{sinx}{1}$

$= tanx$

3. tan^2xcos^2x

$= \frac{sin^2x}{cos^2x}cos^2x$

$= sin^2x$

4. tan^2x

$= \frac{sin^2x}{cos^2x}$

$= \frac{sin^2x}{1-sin^2x}$

5. $1 + \frac{sinx}{cosx}$

$= \frac{cosx}{cosx} + \frac{sinx}{cosx}$

$= \frac{sinx+cosx}{cosx}$

6. $sin^2x + 2cos^2x$

$= sin^2x + cos^2x + cos^2x$

$= 1 + cos^2x$

7. $tanxsinx + cosx$

$= \frac{sinx}{cosx}sinx + cosx$

$= \frac{sin^2x}{cosx} + \frac{cosx}{1}$

$= \frac{sin^2x}{cosx} + \frac{cos^2x}{cosx}$

$= \frac{sin^2x+cos^2x}{cosx}$

$= \frac{1}{cosx}$

Appendix 10: DCAT 2 Activity 2 Questions

Prove the following trigonometric identities:

<u>Day one:</u>

Time: Approximately 40 min

All values of *x* are well defined.

<u>Question 1</u>

Prove that $\frac{1}{tan^2x} + \frac{1}{tanx} = \frac{1+tanx}{tan^2x}$.

Solution:

$\frac{1}{tan^2x} + \frac{1}{tanx}$

$= \frac{1}{tan^2x} + \frac{tanx}{tan^2x}$

$= \frac{1+tanx}{tan^2x}$.

<u>Question 2</u>

Prove that $\frac{1}{sinx} - \frac{1}{tanx} = \frac{1-cosx}{sinx}$.

Solution:

$\frac{1}{sinx} - \frac{1}{tanx}$

$= \frac{1}{sinx} - \frac{1}{\frac{sinx}{cosx}}$

$= \frac{1}{sinx} - \frac{cosx}{sinx}$

$= \frac{1-cosx}{sinx}$.

<u>Question 3</u>

Prove that $sin^4x - cos^4x = 2sin^2x$-1.

Solution:

$sin^4x - cos^4x$

$= (sin^2x - cos^2x)(sin^2x + cos^2x)$

$= \left(sin^2x - (1 - sin^2x)\right)(1)$

$= 2sin^2x - 1.$

<u>Question 4</u>

Prove that $sin^3x + cos^3x = (sinx + cosx)(1 + sinxcosx)$.

Solution:

$sin^3x + cos^3x$

$= (sinx + cosx)(sin^2x - sinxcosx + cos^2x)$

$= (sinx + cosx)(1 - sinxcosx).$

Question 5

Prove that $(tan^2x + 1)(cos^2x - 1) = -tan^2x.$

Solution:

$(tan^2x + 1)(cos^2x - 1)$

$= tan^2xcos^2x - tan^2x + cos^2x - 1$

$= \frac{sin^2x}{cos^2x}cos^2x - tan^2x + cos^2x - 1$

$= sin^2x - tan^2x + cos^2x - 1$

$= -tan^2x + 1 - 1$

$= -tan^2x\ .$

Day two:

Time: Approximately 40 min.

Question 1

Prove that $\frac{7}{2sinx} + \frac{2}{7sinx} = \frac{53}{14sinx}.$

Solution:

$\frac{7}{2sinx} + \frac{2}{7sinx}$

$= \frac{7*7sinx}{2*7sinx} + \frac{2*2sinx}{7*2sinx}$

$= \frac{53sinx}{14sinx}$

$= \frac{53}{14}.$

Question 2

Prove that $\frac{sinx}{cosx} + \frac{cosx}{sinx} = \frac{1}{cosxsinx}.$

Solution:

$\frac{sinx}{cosx} + \frac{cosx}{sinx}$

$= \frac{sin^2x+cos^2x}{sinxcosx}$

$= \frac{1}{sinxcosx}.$

Question 3

Prove that $\frac{15}{3sinxcosx} + \frac{2}{5cosx} = \frac{25+2sinx}{15sinxcosx}$.

Solution:

$\frac{15}{3sinxcosx} + \frac{2}{5cosx}$

$= \frac{5}{sinxcosx} + \frac{2}{5cosx}$

$= \frac{5*5}{5sinxcosx} + \frac{2sinx}{5cosxsinx}$

$= \frac{25+2sinx}{5sinxcosx}$.

Question 4

Prove that $\frac{1}{sinx-cosx} + \frac{1}{sinx+cosx} = \frac{2sinx}{(sinx-cosx)(sinx+cosx)}$.

Solution:

$\frac{1}{sinx - cosx} + \frac{1}{sinx + cosx}$

$= \frac{sinx+cosx}{sinx-cosx} + \frac{sinx-cosx}{sinx+cosx}$

$= \frac{sinx+cosx+sinx-cosx}{(sinx-cosx)(sinx+cosx)}$

$= \frac{2sinx}{(sinx-cosx)(sinx+cosx)}$.

Question 5

Prove that $(1 + \frac{1}{tan^2x})(cos^2x) = \frac{1}{tan^2x}$.

Solution:

$\left(1 + \frac{1}{tan^2x}\right)(cos^2x)$

$= cos^2x + \frac{cos^2x}{tan^2x}$

$= cos^2x + \frac{cos^2x}{\frac{\frac{1}{sin^2x}}{cos^2x}}$

$= cos^2x + \frac{cos^2x}{1}\frac{cos^2x}{sin^2x}$

$= \frac{cos^2x}{sin^2x}\frac{sin^2x}{1} + \frac{cos^2x}{1}\frac{cos^2x}{sin^2x}$

$= \frac{cos^2x(sin^2x+cos^2x)}{sin^2x}$

$= \frac{cos^2x}{sin^2x}$

$= \frac{1}{\frac{sin^2x}{cos^2x}}$

$= \frac{1}{tan^2x}$.

Assessment Two:

Time: 45 minutes.

Please show all your work.

Prove the following trigonometric identities:

1. $\frac{1}{cosx} - \frac{tanx}{1} = \frac{1-sinx}{cosx}$
2. $cos^3x - sin^3x = (cosx - sinx)(1 + cosxsinx)$
3. $tan^2x - \frac{1}{cos^2x} = -1$
4. $tanx + \frac{1}{tanx} = \frac{1}{sinxcosx}$
5. $\frac{5}{sinx} - \frac{5}{cosx} = \frac{5(cosx-sinx)}{sinxcosx}$
6. $\frac{2}{tanx-1} - \frac{2}{tanx+1} = \frac{4}{tan^2x-1}$

Solutions:

1. $\frac{1}{cosx} - \frac{tanx}{1}$

   $= \frac{1}{cosx} - \frac{sinx}{cosx}$

   $= \frac{1-sinx}{cosx}$

2. $cos^3x - sin^3x$

   $= (cosx - sinx)(cos^2x + cosxsinx + sin^2x)$

   $= (cosx - sinx)(1 + cosxsinx)$

3. $tan^2x - \frac{1}{cos^2x}$

   $= \frac{sin^2x}{cos^2x} - \frac{1}{cos^2x}$

   $= \frac{sin^2x-1}{cos^2x}$

   $= \frac{-cos^2x}{cos^2x}$

   $= -1$

4. $tanx + \frac{1}{tanx}$

   $= \frac{sinx}{cosx} + \frac{1}{tanx}$

   $= \frac{sinx}{cosx} + \frac{cosx}{sinx}$

$$= \frac{sin^2x + cos^2x}{sinxcosx}$$

$$= \frac{1}{sinxcosx}$$

5. $\frac{5}{sinx} - \frac{5}{cosx}$

$$= \frac{5cosx}{sinxcosx} - \frac{5sinx}{cosxsinx}$$

$$= \frac{5(cosx - sinx)}{sinxcosx}$$

6. $\frac{2}{tanx - 1} - \frac{2}{tanx + 1}$

$$= \frac{2(tanx + 1)}{tanx - 1} - \frac{2(tanx - 1)}{tanx + 1}$$

$$= \frac{2tanx + 2 - 2tanx + 2}{(tanx - 1)(tanx + 1)}$$

$$= \frac{4}{(tanx - 1)(tanx + 1)}$$

Appendix 11: DCAT 3 Activity 2 Questions

Prove the following trigonometric identities:

Day one:

Time: Approximately 40 min

Question 1

Prove the following trigonometric identity:

$\frac{1}{1-sinx}+\frac{1}{1+sinx}=\frac{2}{cos^2x}$.

Solution:

$$\frac{1}{1-sinx}+\frac{1}{1+sinx}$$

$$=\frac{1+sinx}{(1-sinx)(1+sinx)}+\frac{1-sinx}{(1+sinx)(1-sinx)}$$

$$=\frac{1+sinx+1-sinx}{(1+sinx)(1-sinx)}$$

$$=\frac{2}{1-sin^2x}$$

$$=\frac{2}{cos^2x}.$$

Question 2

Prove the following trigonometric identity:

$1-\frac{cos^2x}{1+sinx}=sinx.$

Solution:

$$1-\frac{cos^2x}{1+sinx}$$

$$=\frac{1+sinx}{1+sinx}-\frac{cos^2x}{1+sinx}$$

$$=\frac{1+sinx-cos^2x}{1+sinx}$$

$$=\frac{1+sinx-(1-sin^2x)}{1+sinx}$$

$$=\frac{1+sinx-1+sin^2x}{1+sinx}$$

$$=\frac{sinx+sin^2x}{1+sinx}$$

$$=\frac{\sin(1+sinx)}{(1+sinx)}$$

$$=sinx.$$

Question 3

Prove the following trigonometric identity:

$\frac{sin^3x+cos^3x}{sinx+cosx} = 1 - sinxcosx.$

Solution:

$\frac{sin^3x + cos^3x}{sinx + cosx}$

$= \frac{(sinx+cosx)(sin^2x-sinxcosx+cos^2x)}{sinx+cosx}$

$= \frac{sin^2x-sinxcosx+cos^2x}{1}$

$= 1 - sinxcosx.$

Question 4

Prove the following trigonometric identity:

$\frac{tanx}{1-tan^2x} = \frac{sinxcosx}{cos^2x-sin^2x}.$

Solution:

$\frac{tanx}{1 - tan^2x}$

$= \frac{\frac{sinx}{cosx}}{\frac{cos^2x}{cos^2x}-\frac{sin^2x}{cos^2x}}$

$= \frac{\frac{sinx}{cosx}}{\frac{cos^2x-sin^2x}{cos^2x}}$

$= \frac{sinx}{cosx}\frac{cos^2x}{cos^2x-sin^2x}$

$= \frac{sinxcosx}{cos^2x-sin^2x}.$

Day two:

Proving trigonometric identities.

Time: Approximately 40 minutes

Question 1

Prove the following trigonometric identity:

$tanx - \frac{1}{tanx} = \frac{1-2cos^2x}{sinxcosx}.$

Solution:

$tanx - \frac{1}{tanx}$

$= \frac{sinx}{cosx} - \frac{cosx}{sinx}$

$= \frac{sin^2-cos^2x}{sinxcosx}$

$= \frac{1-cos^2x-cos^2x}{sinxcosx}$

$= \frac{1-2cos^2x}{sinxcosx}$

Question 2

Prove the following trigonometric identity:

$\frac{sin^2x+3sinx+2}{3sin^2x+7sinx+2} = \frac{sinx+1}{3sinx+1}$.

Solution:

$$\frac{sin^2x + 3x + 2}{3sin^2x + 7sinx + 2}$$

$= \frac{(sinx+1)(sinx+2)}{(3sinx+1)(sinx+2)}$

$= \frac{sinx+1}{3sinx+1}$

Question 3

Prove the following trigonometric identity:

$\frac{1}{1+sinx} + \frac{1}{1-sinx} = \frac{2}{cos^2x}$.

Solution:

$$\frac{1}{1 + sinx} + \frac{1}{1 - sinx}$$

$= \frac{1-sinx}{(1+sinx)(1-sinx)} + \frac{1+sinx}{(1-sinx)(1+sinx)}$

$= \frac{1-sinx+1+sinx}{(1-sinx)(1+sinx)}$

$= \frac{2}{1-sin^2x}$

$= \frac{2}{cos^2x}$

Question 4

Prove the following trigonometric identity:

$\frac{1}{cosx+1} + \frac{1}{cosx-1} = -2\frac{cosx}{sin^2x}$.

Solution:

$$\frac{1}{cosx + 1} + \frac{1}{cosx - 1}$$

$= \frac{cosx-1}{(cosx+1)(cosx-1)} + \frac{cosx+1}{(cosx-1)(cosx+1)}$

$= \frac{cosx-1+cosx+1}{cos^2x-1}$

$= \frac{2cosx}{-sin^2x}$.

<u>Assessment Three:</u>

Prove the following trigonometric identities. Show all your work.

Time: 45 minutes.

1. $\frac{1}{1+cosx} + \frac{1}{1-cosx} = \frac{2}{sin^2x}$
2. $\frac{1}{cos^2x} + \frac{1}{sin^2x} = \frac{1}{cos^2x}\frac{1}{sin^2x}$
3. $1 - \frac{sin^2x}{1+cosx} = cosx$
4. $\frac{sin^2x-cos^2x}{sinx-cosx} = sinx + cosx$
5. $\frac{sin^2x+4sinx-5}{sin^2x+sinx-2} = \frac{sinx+5}{sinx+2}$

Solutions:

1. $\frac{1}{1+cosx} + \frac{1}{1-cosx}$

$= \frac{1-cosx}{(1+cosx)(1-cosx)} + \frac{1+cosx}{(1-cosx)(1+cosx)}$

$= \frac{1-cosx+1+cosx}{1-cos^2x}$

$= \frac{2}{sin^2x}$

2. $\frac{1}{cos^2x} + \frac{1}{sin^2x}$

$= \frac{sin^2x+cos^2x}{cos^2xsin^2x}$

$= \frac{1}{cos^2x}\frac{1}{sin^2x}$

3. $1 - \frac{sin^2x}{1+cosx}$

$= \frac{1+cosx}{1+cosx} - \frac{sin^2x}{1+cosx}$

$= \frac{1+cosx}{1+cosx} - \frac{1-cos^2x}{1+cosx}$

$= \frac{1+cosx-1+cos^2x}{1+cosx}$

$= \frac{cosx+cos^2x}{1+cosx}$

$= \frac{cosx(1+cosx)}{1+cosx}$

$= cosx$

4. $\frac{sin^2x-cos^2x}{sinx-cosx}$

$= \frac{(sinx-cosx)(sinx+cosx)}{sinx-cosx}$

$= sinx + cosx$

5. $\frac{sin^2x+4sinx-5}{sin^2x+sinx-2}$

$= \frac{(sinx+5)(sinx-1)}{(sinx+2)(sinx-1)}$

$= \frac{sinx+5}{sinx+2}$

Appendix 12: Explanation of DCAT 1 Activity 1 questions

During the first two days of this study, all questions were on the DCAT 1 level relating to the simplification of trigonometric expressions. Questions 1 to 5 were completed on day one. Questions 6 to 10 were completed on day two.

Question 1

Simplify the trigonometric expression $sinx\frac{1}{tanx}$.

The teacher used this question to explain what is meant by '*simplification'*. That is, he replaced the *trigonometric identity* for $tanx = \frac{sinx}{cosx}$ and simplified the expression. The concept of a fraction divided by a fraction was reaffirmed by the teacher.

Question 2

Simplify the trigonometric expression $sin^2x + cos^2x + 1$.

For this question the teacher showed the students how they can use the *trigonometric identity* $sin^2x + cos^2x = 1$ to simplify the trigonometric expression. This question does not contain the trigonometric ratio of $tanx$ as the previous one does.

Question 3

Simplify the trigonometric expression $sinxtanxcosx$.

The main purpose of this question was to ease the students into the process of simplification and build their confidence. The students could easily relate this question to the one that the teacher explained in question 1.

Question 4

Simplify the trigonometric expression $(\frac{cosx}{tanx})sinx$.

Unlike the previous questions, this question contained a rational trigonometric expression with the denominator *not equal to 1.* The purpose of the question was to see how the students would deal with the 'fraction divided by a fraction' concept.

Question 5

Simplify the trigonometric expression $\frac{sinx+cosx}{sinxcosx}$.

This question tested whether the student knew how to apply $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$. The previous question did not contain addition or subtraction.

Question 6

Simplify the trigonometric expression $\frac{sinx}{tanx}$.

Although this is Question 6, it is also the first question of the second day of this study. It seemed appropriate to start with an easy question. This question addressed the notion of a fraction divided by 'a fraction divided by a faction'.

Question 7

Simplify the trigonometric expression $(sinx + cosx)^2 - 2sinxcosx$.

This question addressed the expansion of a binomial squared as well as the application of $sin^2x + cos^2x = 1$.

Question 8

Simplify the trigonometric expression tan^2xcos^2x.

Unlike the previous question, this question addressed whether students understood that $tan^2x = tanx * tanx$ and that $\left(\frac{sinx}{cosx}\right)^2 = \frac{sinx}{cosx} * \frac{sinx}{cosx}$.

Question 9

Simplify the trigonometric expression $tan^2x + 1$.

In the previous question, the students established that $tan^2x = \frac{sin^2x}{cos^2x}$. However, this question tested whether the students could write 1 as $\frac{cos^2x}{cos^2x}$.

Question 10

Simplify the trigonometric expression $\frac{cosx}{1-sin^2x}$.

This question addressed whether the students could identify whether $1 - sin^2x$ is related to $sin^2x + cos^2x = 1$ and whether it would be appropriate to cancel.

Appendix 13: Explanation of DCAT 2 Activity 1 questions

Question 1

Simplify the trigonometric expression $\frac{sinx}{cos^3x} \div \frac{tanx}{cos^3x}$**.**

This question encompassed elements that were established under the Basic Abilities Level, namely a fraction divided by a fraction; cancellation; and writing $tanx\ as\ \frac{sinx}{cosx}$.

Question 2

Simplify the trigonometric expression $sinxcos^2x - sinx$.

This question incorporated *factorisation* and a trigonometric identity. This was the first time that *factorisation* was incorporated into a question.

Question 3

Simplify the trigonometric expression $\frac{2cosx}{sinx} + \frac{2sinx}{cosx}$**.**

This was the first time that the students' understanding of adding fractions was tested. This question also incorporated a trigonometric identity.

Question 4

Simplify the trigonometric expression $\frac{cos^2x}{1-sinx}$**.**

This question incorporated two concepts. One, rewriting $cos^2x = 1 - sin^2x$; and two, recognising that $1 - sin^2x$ can be factorised as the difference between two squares.

Question 5

Simplify the trigonometric expression $sinx + \frac{cosx}{tanx}$**.**

This question differs from the previous question in that it incorporates most of the concepts mentioned previously, namely a fraction divided by a fraction, adding two trigonometric fractions, identifying an identity and cancellation.

Question 6

Simplify the trigonometric expression $sinxcosxtanx + cos^2x$**.**

In this question sin^2x was embedded in $sinxcosxtanx$, which when added to cos^2x, yielded 1.

Question 7

Simplify the trigonometric expression $\frac{1}{sin^2x}+\frac{1}{cos^2x}$**.**

Unlike Question 6 where sin^2x was embedded in $sinxcosxtanx$, in this question sin^2x and cos^2x were embedded in the numerator.

Question 8

Simplify the trigonometric expression $\frac{sin^2x}{1+cosx}$**.**

This question was similar to Question 4. However, in this question sin^2x must be converted to the difference between two squares, factorised, and then cancelled.

Question 9

Simplify the trigonometric expression $\frac{1}{tan^2x+1}$**.**

This question was different from the previous ones in the sense that tan^2x+1 should be converted to a monomial.

Question 10

Simply the trigonometric expression $\frac{sin^2x+3sinx+2}{sin^2x+4sinx+3}$**.**

This question assessed the students' ability to identify and factorise a quadratic expression.

Appendix 14: Explanation of DCAT 3 Activity 1 questions

The next eight Questions were on the DCAT 3 level.

Question 1

Simplify the trigonometric expression $tan^2x - tan^2xsin^2x$.

In order to solve this question, the student should have realised that they first needed to take out the *common factor*, then replace *two trigonometric identities*, and then *simplify* by *cancelling.*

Question 2

Simplify the trigonometric expression $\frac{2sin^2x - sinx - 1}{sinx - 1}$.

Unlike the previous question, this question required the students to realise that the numerator needs *quadratic factorisation.* The question also tested whether the students knew how and when (or when not) to *simplify.*

Question 3

Simplify the trigonometric expression $tanx + \frac{cosx}{1+sinx}$.

This was the first question where one of the denominators was not a monomial.

Question 4

Simplify the trigonometric expression $\frac{sinx - cosx}{sinx + cosx} + \frac{sinx + cosx}{sinx - cosx}$.

In this question, unlike the previous one, both *denominators consisted of two terms.*

Question 5

Simplify the trigonometric expression $\frac{sin^3x + cos^3x}{sinx + cosx}$

This was the first question that examined whether the students could *identify a*nd *factorise the sum of cubes,* and make the correct choice of *cancelling.*

Question 6

Simplify the trigonometric expression $\frac{2sinx + 4}{3sin^2x + 7sinx + 2}$.

Unlike the previous question, this question examined the students' understanding of *quadratics* and *cancelling.*

Question 7

Simplify the trigonometric expression $\frac{1}{sin^2x-1}+\frac{sinx}{sin^2x-1}$**.**

Although initially the question seemed 'easy' since there is a common denominator, the students should have realised the need to *factorise the denominator as the difference between two squares,* and then perform *cancellation.*

Question 8

Simplify the trigonometric expression $\frac{1-cos^2x}{sin^2x}-\frac{1-sin^2x}{cos^2x}$.

This question tested whether students understood that each of the numerators was the trigonometric identity of its denominator.

Appendix 15: Explanation of DCAT 1 Activity 2 questions – Proving of trigonometric identities

During the first two days, all questions were at the DCAT 1 level. Questions 1 to 5 were completed on day one. Questions 6 to 10 were completed on day two.

Question 1

Prove that $\boldsymbol{cosxtanx = sinx}$

This question tested whether the students could rewrite $tanx$ and then perform cancellation.

Question 2

Prove that $\boldsymbol{1 - sin^2x + cos^2x = 2cos^2x}$.

This question tested whether the students recalled that $\boldsymbol{sin^2x + cos^2x = 1}$.

Question 3

Prove that $\boldsymbol{\frac{\frac{1}{cosx}}{\frac{1}{sinx}} = tanx}$.

The concept of 'fraction divided by a fraction' was tested with this question.

Question 4

Prove that $\boldsymbol{16sin^2xcosx + 12cos^2xsinx = 4sinxcosx(4sinx + 3cosx)}$.

Contrasting with the previous question, this question tested whether the students could either *apply the distributive property* or *factorise out a common factor.*

Question 5

Prove that $\boldsymbol{\frac{1}{sin^2x}\frac{1}{cos^2x} = \frac{1}{sin^2x} + \frac{1}{cos^2x}}$.

This question was used to show that an identity could be proven starting on *either side of an equation.* This was the first identity that used the addition of fractions coupled with identifying that $\boldsymbol{sin^2x + cos^2x = 1}$

For the next five questions, the teacher cut out the steps required to answer each question and scrambled them. He then handed the scrambled pieces to the students to unscramble and logically arrange each proof.

Question 6

Prove that $\boldsymbol{tanx + \frac{2}{tanx} = \frac{sin^2x+2cos^2x}{sinxcosx}}$ **.**

The quickest way to prove this identity was to start on the *right-hand side of the equation* using the algebraic property $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ and then perform cancellation.

Question 7

Prove that $\boldsymbol{tanx + \frac{1}{tanx} = \frac{1}{sinxcosx}}$**.**

Though similar to the previous question, this differs in that it was easier to prove the identity by starting on the *left-hand side of the equation.*

Question 8

Prove that $\boldsymbol{\frac{tanx-sinx}{sinx} = \frac{1}{cosx} - 1}$**.**

This question used the same technique employed for Question 6, but was best solved by starting on the *left-hand side of the equation.*

Question 9

Prove that $\boldsymbol{\frac{1}{cos^2x}\left(1 - cos^2x\right) = tan^2x}$**.**

This question differs from the previous question in that the students should have replaced $\boldsymbol{\left(1 - cos^2x\right)}$ first, followed by the replacement of $\frac{sin^2x}{cos^2x}$.

Question 10

Prove that $\boldsymbol{sinx - sinxcos^2x = sin^3}x$.

This is the first question in which the students should have factorised out a common factor, and then replaced $1 - cos^2x$ with sin^2x.

Appendix 16: Explanation of DCAT 2 Activity 2 questions

The next set of 10 questions was based on the DCAT 2 level. Questions 1 to 5 were completed on day one. Questions 6 to 10 were completed on day two.

Question 1

Prove that $\frac{1}{tan^2x}+\frac{1}{tanx}=\frac{1+tanx}{tan^2x}$.

This question was designed so that the student could use the algebraic property $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$ and start on the right-hand side of the equation, ***or*** use the procedures for adding algebraic fractions and start on the left-hand side of the equation.

Question 2

Prove that $\frac{1}{sinx}-\frac{1}{tanx}=\frac{1-cosx}{sinx}$.

Unlike the previous question, this question was easily solved by applying the algebraic property $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$ to the right-hand side of the equation.

Question 3

Prove that $sin^4x-cos^4x=2sin^2x\text{-}1$.

This question required the students to factorise using the *difference between two squares* twice.

Question 4

Prove that $sin^3x+cos^3x=(sinx+cosx)(1+sinxcosx)$.

Unlike the previous question which required prior knowledge of the *difference between two squares,* this question required knowledge of the *sum of two cubes.*

Question 5

Prove that $(tan^2x+1)(cos^2x-1)=-tan^2x$.

This question required students to apply the *distributive property* and *did not require factorisation* as the previous question did.

Question 6

Prove that $\frac{7}{2sinx}+\frac{2}{7sinx}=\frac{53}{14}$.

This question required the students to start the proof on the *left-hand side* of the equation.

Question 7

Prove that $\frac{sinx}{cosx} + \frac{cosx}{sinx} = \frac{1}{cosxsinx}$.

If students wished to start the question on the *right-hand side* of the equation, they required prior knowledge of the *partial decomposition of fractions***.** However, starting on the *left-hand* side required the ability to add algebraic fractions.

Question 8

Prove that $\frac{15}{3sinxcosx} + \frac{2}{5cosx} = \frac{25+2sinx}{15sinxcosx}$.

This question required the students to *reduce or simplify the left-hand side of the equation* before attempting to solve the question.

Question 9

Prove that $\frac{1}{sinx-cosx} + \frac{1}{sinx+cosx} = \frac{2sinx}{(sinx-cosx)(sinx+cosx)}$.

This question was designed to give students practice in working with binomial denominators.

Question 10

Prove that $(1 + \frac{1}{tan^2x})(cos^2x) = \frac{1}{tan^2x}$.

To prove this identity, the students had to apply the *distributive law* first and then *add two algebraic fractions.*

Appendix 17: Explanation of DCAT 3 Activity 2 questions

The next set of eight questions was based on the DCAT 3 level. Questions 1 to 4 were completed on day one. Questions 5 to 8 were completed on day two. The number of questions is lower than for the other DCAT levels because of their level of difficulty.

Question 1

Prove that $\frac{1}{1-sinx} + \frac{1}{1+sinx} = \frac{2}{cos^2x}$.

This question required the students to add two fractions. Next, the students were required to convert the denominator using a trigonometric identity.

Question 2

Prove that $1 - \frac{cos^2x}{1+sinx} = sinx$

Unlike in the previous question, the students were required to write 1 as $\frac{1+sinx}{1+sinx}$. Furthermore, they should have used a trigonometric identity, factorised, and performed cancelling.

Question 3

Prove that $\frac{sin^3x+cos^3x}{sinx+cosx} = 1 - sinxcosx.$

In contrast with the previous question, here the students were required to apply the sum of cubes factorisation and use a trigonometric identity.

Question 4

Prove that $\frac{tanx}{1-tan^2x} = \frac{sinxcosx}{cos^2x-sin^2x}$.

This was the first question in which the students had to consider each of the denominators as the sum of two fractions.

Question 5

Prove that $tanx - \frac{1}{tanx} = \frac{1-2cos^2x}{sinxcosx}$.

This question required the students to start on the left-hand side of the equation, replacing $tanx$ with $\frac{sinx}{cosx}$.

Question 6

Prove that $\frac{sin^2x+3sinx+2}{3sin^2x+7sinx+2} = \frac{sinx+1}{3sinx+1}$.

The students should have recognised that both the numerator and the denominator of the left-hand side of the equation were *factorisable quadratics.*

Question 7

Prove that $\frac{1}{1+sinx} + \frac{1}{1-sinx} = \frac{2}{cos^2x}$.

Unlike in the previous question, the students were required to replace the denominator with a trigonometric identity.

Question 8

Prove that $\frac{1}{cosx+1} + \frac{1}{cosx-1} = -2\frac{cosx}{sin^2x}$.

Unlike the previous question, in which the students could have started on the left-hand side of the equation, this question gave the students the option to start on *either side* of the equation.

Appendix 18: Focus group interview guideline

The standardised open interview style arose from the fact that the same question was posed to each of the participants in the focus group. The starting question of the focus group interview was always, “How did things go today? What was difficult, confusing or frustrating?” The informal conversational interview style was adopted so the researcher could listen with an open mind and follow where the students’ arguments led. Probing questions were used to delve deeper into an issue that might occur during the interview. For example, a student might say, “I feel that the teacher did not explain the concept clearly enough.” The researcher would then prompt the student to elaborate, by asking, “What is still unclear to you about the concept?” This type of interview style is in line with Dilshad and Latif (2013), who regarded the focus group interview as a tool for qualitative research.